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UPPER BOUNDS AND SAINT-VENANT'S PRINCIPLE IN TRANSIENT HEAT CONDUCTION*

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Summary. An investigation is carried out on transient heat-conduction problems with prescribed surface temperature, and the validity of Saint-Venant's principle in parabolic boundary-value problems is discussed.

Some upper bounds for the steady-state temperature in a body whose surface temperature is prescribed over a small portion of the bounding surface were derived in a recent investigation [1] concerned with various possible formulations of Saint-Venant's principle. It was indicated there that this principle was a general property of elliptic differential equations, and would not hold all the time for differential equations of other types. For the parabolic type, however, it appeared that such a principle could be valid in a rather general form; this conclusion was borne out by earlier work [2] in which actual use of the principle was made. It is the purpose of this work to investigate this matter a little further, and to establish for the transient heat conduction case (which is of the parabolic type) the same sort of upper bounds which were treated in [1].

We note first a number of properties of fundamental solutions of the Fourier heat-conduction equation

$$\kappa\nabla^2 T(Q, t) = \frac{\partial T(Q, t)}{\partial t} \tag{1}$$

which follow from the fact that [3]**

$$\text{if} \quad T(P, t) \geq 0 \quad \text{and} \quad T(Q, 0) = 0, \quad \text{then} \quad T(Q, t) \geq 0 \tag{2}$$

or in other words that it is impossible to lower the temperature anywhere in the interior of a body by raising its boundary value.

Consider now the fundamental solution corresponding to a unit source liberated at Q_2 at a time τ and with vanishing temperature on the surface (analogous therefore to a Green's function); the temperature at Q_1 due to this source is denoted by $G(Q_1, Q_2, t - \tau)$, and then

$$G(P, Q_2, t - \tau) = 0; \qquad G(Q_1, Q_2, 0) = 0. \tag{3}$$

The following properties can now be derived:

(a) $G(Q_1, Q_2, t - \tau) \geq 0$, since Eqs. (3) hold, and G is positive by definition in the neighborhood of $Q_1 = Q_2$, and regular everywhere but at $Q_1 = Q_2$; therefore Eqs. (2) apply and give immediately the desired result.

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**We will always use P to denote a point on the boundary, and Q a generic point, either on the boundary or in the interior.

(b) It follows from (a) that

$$\frac{\partial G(P, Q_2, t - \tau)}{\partial n_p} \geq 0$$

where n_p is the inward normal to the surface at P.

(c) Consider two domains, D and D^*, where D is completely contained in D^*, and let G and G^* be the corresponding Green's functions; then $G^*(Q_1, Q_2, t - \tau) \geq G(Q_1, Q_2, t - \tau)$ with Q_1 and Q_2 in D. The difference $(G^* - G)$ is in fact initially zero for Q_1 in D, and its value on the surface of D is G^* because of (a); hence Eqs. (3) apply and the desired result follows.

(d) Consider now the case in which the surfaces of the two domains just described have a common portion; it then follows from (c) that on the common boundary

$$\frac{\partial G^*(P, Q_2, t - \tau)}{\partial n_p} \geq \frac{\partial G(P, Q_2, t - \tau)}{\partial n_p}.$$

Note that all the above properties are well-known in the steady state case [4].

The solution due to a prescribed surface temperature is [5]:

$$T(Q, t) = \kappa \int_0^t \int_S T(P, \tau) \frac{\partial G(P, Q, t - \tau)}{\partial n_p} dS(P)\, d\tau, \tag{4}$$

where S is the surface of the body, and where it has been assumed that $T(Q, 0) = 0$. We wish to consider a problem in which $T(P, t)$ is zero everywhere except in a (small) portion S_0, where we may set

$$| T(P, t) | \leq T_1(t) \leq T_0 . \tag{5}$$

Hence with property (b) we have

$$| T(Q, t) | \leq \kappa \int_0^t T_1(\tau) \int_{S_0} \frac{\partial G(P, Q, t - \tau)}{\partial n_p} dS(P)\, d\tau \tag{6}$$

and, further, if D and D^* are two domains related as in (d) above:

$$| T(Q, t) | \leq \kappa \int_0^t T_1(\tau) \int_{S_0} \frac{\partial G^*(P, Q, t - \tau)}{\partial n_p} dS(P)\, d\tau \tag{7}$$

Eqs. (6) and (7) represent upper bounds to the solution. The determination of G or G^* is, however, often very difficult [6], but, as in [1], it may be at times possible to take for D^* a half-space (say $x > 0$ in Cartesian coordinates); then

$$G^*(Q_1, Q_2, t - \tau) = \frac{1}{8[\pi\kappa(t - \tau)]^{3/2}} \left\{ \exp\left[-\frac{R^2}{4\kappa(t - \tau)} \right] - \exp\left[-\frac{R^{*2}}{4\kappa(t - \tau)} \right] \right\} \tag{8}$$

where R is the distance between Q_1 and Q_2, and R^* the distance between Q_1 and the reflection of Q_2 in the plane $x = 0$. With the aid of (8) the integrations in Eqs. (6) and (7) can be carried out; for example, if S_0 is the surface bounded by the circle $y^2 + z^2 = h^2/\pi$, whose area is h^2, then the upper bound along the line $y = z = 0$ is given by

$$\frac{| T(Q, t) |}{T_0} \leq \operatorname{erf}\left[\frac{x}{(4\kappa t)^{1/2}} \right] - \left(1 + \frac{h^2}{\pi x^2} \right)^{-1/2} \operatorname{erf}\left[\frac{x}{(4\kappa t)^{1/2}} \left(1 + \frac{h^2}{\pi x^2} \right) \right], \tag{9}$$

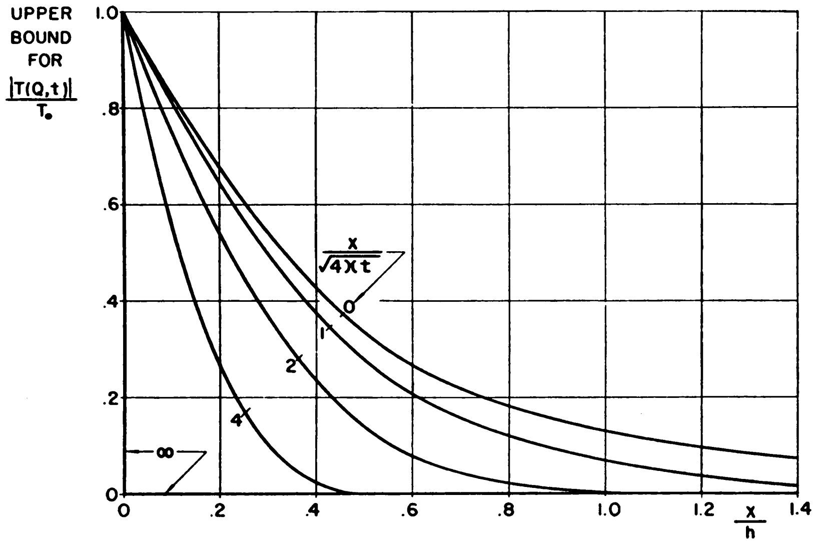

FIG. 1

where the constant T_0 has been used in place of the function $T_1(t)$ so as to avoid the choice of specific applied-temperature history. The right-hand side of (9), a function of the two parameters (x/h) and $[h/(4\,\kappa\,t)^{\frac{1}{2}}]$, is plotted in Fig. 1. Note that as $t \to \infty$ the steady value for the upper bound is reached, i.e. $1 - [1 + (h^2/\pi x^2)]^{-\frac{1}{2}}$ as found in Ref. [2]*. For $(x/h) = 0$ the upper bound is always unity, and as $(x/h) \to \infty$ it always approaches zero; this approach is quite rapid for all values of the parameter $h/(4\,\kappa\,t)^{\frac{1}{2}}$, so that the temperature for $(x/h) > 1$ could possibly be considered small compared to T_0. In this case we may then say that Saint-Venant's Principle holds, and it is of interest to note that it holds more strongly for the transient than for the steady state, or, in other words that the steady state values provide an upper bound for the transient cases, in agreement with [2]. This type of conclusion cannot be extended to problems in which the transient case is governed by a hyperbolic equation [7].

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*Except for a numerical error in that reference which however in no way alters the conclusions reached.