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CERTAIN SOLUTIONS OF THE HEAT CONDUCTION EQUATION*

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1. Introduction. In the following we consider solutions of the heat conduction equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}, \qquad k = K/\rho c, \tag{1.1}$$

for $x > 0$, $t > 0$, corresponding to certain heat inputs $h(t)$ for $t > 0$ over the plane $x = 0$: initially T vanishes for $x > 0$. In (1.1) ρc is the specific heat per unit volume, K the conductivity.

To this end we start with the "Green's function" or the "instantaneous heat source" solution

$$G(x, t) = \begin{cases} \dfrac{\exp[-x^2/4kt]}{2(\pi kt)^{1/2}}, & t > 0, \\ 0, & t < 0. \end{cases} \tag{1.2}$$

The function G satisfies Eq. (1.1) for $t > 0$ and represents the temperature due to an amount of heat discharged at the time $t = 0$ at $x = 0$, in a medium of initial temperature $T = 0$, the quantity of heat per unit area of the plane $x = 0$ being such that

$$\int_{-\infty}^{\infty} G(x, t)\, dx = 1, \qquad t > 0. \tag{1.3}$$

The function G is Gaussian in x for each $t > 0$ and has a deviation varying as $t^{1/2}$. For $x = 0$, $t > 0$, G varies as $t^{-1/2}$. At $x = 0$, $t = 0$, G possesses a singularity.

Assume that in a semi-infinite medium $x > 0$, initially at $T = 0$, heat of amount $h(t)$ is fed in at $x = 0$ for $t > 0$. The temperature is given for $t > 0$ by the following definite integral:

$$\begin{aligned} T(x, t) &= \frac{2}{\rho c} \int_0^t h(t') G(x, t - t')\, dt' \\ &= \frac{1}{\rho c} \int_0^t h(t') \frac{\exp[-x^2/4k(t - t')]}{[\pi k(t - t')]^{1/2}}\, dt'. \end{aligned} \tag{1.4}$$

The factor ρc in (1.4) is due to the specific heat of the material per unit volume: the factor 2 in the first integral is due to the fact that in (1.2) the heat flows to *both* sides of $x = 0$, while $h(t)$ is defined as the heat flowing only to the side $x > 0$.

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For $x = 0$, Eq. (1.4) yields

$$T(0, t) = \frac{1}{\rho c} \int_0^t \frac{h(t')\, dt'}{[\pi k(t - t')]^{1/2}}. \tag{1.5}$$

In particular, let

$$h(t) = \begin{cases} t^n/n! = t^n/\Gamma(n+1), & t > 0, \quad n > -1. \\ 0, & t < 0. \end{cases} \tag{1.6}$$

Upon introducing the variable of integration

$$t' = tu, \tag{1.7}$$

Eq. (1.6) may be reduced to the beta-integral, yielding

$$T(0, t) = \frac{t^{n+1/2}\Gamma(1/2)}{\rho c(\pi k)^{1/2}\Gamma(n + 3/2)} = \frac{t^{n+1/2}}{(\rho c K \pi)^{1/2}} \cdot \frac{1}{(1/2) \cdots (n - 1/2)(n + 1/2)}. \tag{1.8}$$

Equations (1.6), (1.8) are valid even for fractional $n > -1/2$, provided $n!$ is interpreted as $\Gamma\,(n + 1)$.

The explicit expression (1.8) for $T(0, t)$ can be applied to general $h(t)$ by approximating to the latter by means of a polynomial in t

$$h(t) = h_0 + h_1 t + h_2 t^2/2! + \cdots + h_n t^n/n! \tag{1.9}$$

and carrying out the corresponding superposition of the solutions (1.8)

$$T(0, t) = \frac{t^{1/2}}{\rho c(\pi k)^{1/2}} \left[\frac{h_0}{(1/2)} + \frac{h_1 t}{(1/2)(3/2)} + \frac{h_2 t^2}{(1/2)(3/2)(5/2)} + \cdots + \frac{h_n t^n}{(1/2) \cdots (n + 1/2)} \right]. \tag{1.10}$$

Turning to the integration of (1.4) for general x, it is shown in Sec. 2 that for the heat input (1.6) for $n = 0, 1, 2, \cdots$ the resulting temperature is given by

$$T = T_n(x, t) = \frac{t^{n+1/2}}{(\rho c K)^{1/2}} f_n(u), \qquad u = \frac{x}{2(kt)^{1/2}}, \tag{1.11}$$

TABLE I

n	P_n	Q_n
-1	$1/2$	0
0	1	$-2u$
1	$2(1 + u^2)/3$	$-2u - 4u^3/3$
2	$(4 + 9u^2 + 2u^4)/15$	$-u + 4u^3/3 + 4u^5/15$

where f_n, as indicated, depends only on u, and is given by

$$f_n(u) = 2\pi^{-1/2}P_n(u) \exp(-u^2) + Q_n(u) \operatorname{erfc}(u) , \tag{1.12}$$

where $P_n(u)$, $Q_n(u)/u$ are certain polynomials in u^2 of degree n, and "erfc" denotes the "complementary error function." For $n = 0, 1, 2$, P_n, Q_n are given in Table I. The row $n = -1$ in Table I is explained in Sec. 2, where recurrence equations for P_n, Q_n are also given, as well as expansions for T_n in powers of x.

Solutions for T_n for *non-integer n* are discussed in Sec. 3, where operational expressions for T_n are also given. It is shown that these solutions of (1.1) can be extended to $x < 0$ and correspond to proper initial temperatures which vanish for $x > 0$.

2. Solutions for polynomial power inputs. We consider heat inputs at $x = 0$, of the form (1.6) for integer n

$$h(t) = h_n(t) = 1t^n/n!, \tag{2.1}$$

where $1 = H(t)$ is the Heaviside unit function defined by

$$1(t) = H(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0. \end{cases} \tag{2.2}$$

It will be noted that h_n satisfy the relations

$$\frac{dh_n(t)}{dt} = h_{n-1}(t). \tag{2.3}$$

Therefore the corresponding temperatures $T_n(x, t)$ will satisfy similar relations

$$\frac{\partial T_n(x, t)}{\partial t} = T_{n-1}(x, t), \qquad T_n(x, t) = 0 \quad \text{for} \quad t < 0. \tag{2.4}$$

The sequence $h_n(t)$, $T_n(t)$ may be extended by means of (2.3), (2.4), but not directly by means of (2.1), to $n = -1$, yielding

$$h_{-1}(t) = \frac{dh_0(t)}{dt} = \frac{dH(t)}{dt} = \delta(t), \tag{2.5}$$

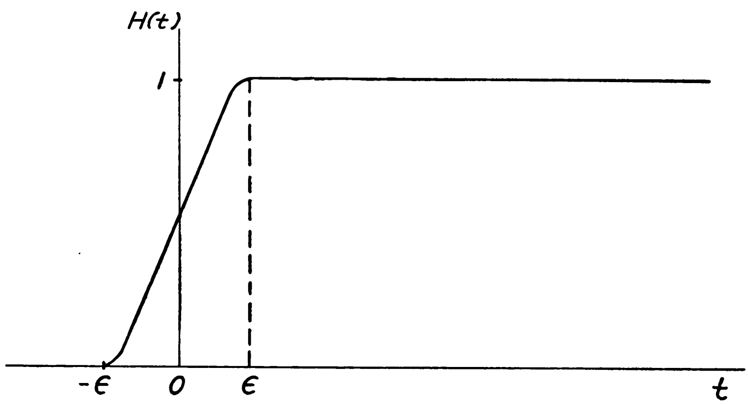

FIG. 2-1

where $\delta(t)$ denotes the "unit impulse function", or the "Dirac function". Indeed, if $H(t)$ be approximated by means of an analytic curve as in Fig. 2-1, then its slope will take on the appearance shown in Fig. 2-2, showing a hump of unit area near $t = 0$. In the limit, as $\epsilon \to 0$, there results an instantaneous heat input for which the temperature is given, except for a factor $2/\rho c$, by Eq. (1.2), namely,

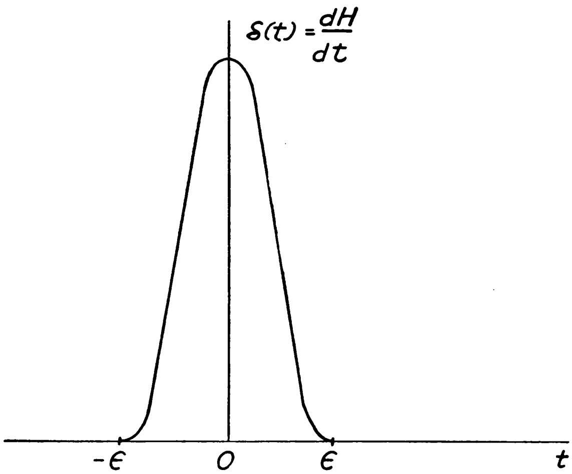

FIG. 2-2

$$T_{-1}(x, t) = \frac{2}{\rho c} G(x, t) = \begin{cases} \dfrac{t^{-1/2}}{(\pi \rho c K)^{1/2}} \exp(-u)^2, & u = \dfrac{x}{2(kt)^{1/2}}, \quad t > 0, \\ 0, \quad t < 0. \end{cases} \tag{2.6}$$

It will be noted that Eq. (2.6) agrees with Eqs. (1.11), (1.12), provided P_n, Q_n are chosen as in Table I for $n = -1$.

For $n = 0$, when (2.1) yields $h(t) = 1$ for $t > 0$, the temperature T_0 may be calculated from (2.4), (2.6) or from (1.4). Carrying out the integration by parts, one obtains

$$T_0(x, t) = \left(\frac{t}{\rho c K}\right)^{1/2} \left[\frac{2 \exp(-u^2)}{\pi^{1/2}} - 2u \operatorname{erfc} u\right]. \tag{2.7}$$

The corresponding integrations (1.4) or (2.4) have been carried out for T_n for $n = 1, 2$. The results suggest for general integer n, the form (1.11), (1.12). Indeed, substitution of (1.11) in (2.4) verifies the assumption (1.11) provided the recurrence equations

$$[(n + 1/2)f_n(u) - (u/2)f_n'(u)] = f_{n-1}(1) \tag{2.8}$$

are satisfied. Multiplying both sides by $2/u^{(2n+2)}$, there results

$$\frac{(2n + 1)f_n(u)}{u^{2n+2}} - \frac{f_n'(u)}{u^{2n+1}} = \frac{2f_{n-1}(u)}{u^{2n+2}}, \tag{2.9}$$

where the left side is the derivative of $-f_n(u)/u^{2n+1}$. Hence,

$$f_n(u) = -2u^{2n+1} \int_{u_0}^{u} \frac{f_{n-1}(u)\, du}{u^{2n+2}} + Cu^{2n+1}, \tag{2.10}$$

where C is a constant. In view of the condition $T_n \to 0$ for $x \to \infty$ or $t \to 0$, the choices $u_0 = +\infty$, $C = 0$, are proper. One obtains

$$f_n(u) = 2u^{2n+1} \int_u^\infty \frac{f_{n-1}(u)\,du}{u^{2n+2}}. \tag{2.11}$$

For $n = -1$ Eq. (2.6) yields

$$f_{-1}(u) = \exp(-u^2)/\pi^{1/2}. \tag{2.12}$$

Hence, Eq. (2.11) now leads to

$$f_0(u) = \frac{2u}{\pi^{1/2}} \int_u^\infty \frac{\exp(-u^2)}{u^2}\,du. \tag{2.13}$$

Integration by parts again leads to (2.7).

Equation (2.7) is of the form (1.11), (1.12) with

$$P_0 = 1, \qquad Q_0 = -2u. \tag{2.14}$$

Applying (2.11) for $n = 1$ yields f_1 of the form (1.12) with

$$P_1 = \frac{2}{3}(1 + u^2), \qquad Q_1 = -2u - \frac{4}{3}u^3. \tag{2.15}$$

A similar calculation for $n = 2$ shows that T_2 is given by (1.11), (1.12) with

$$P_2 = \frac{4}{15} + \frac{3}{5}u^2 + \frac{2}{15}u^4, \qquad Q_2 = -\left[u + \frac{4}{3}u^3 + \frac{4}{15}u^5\right]. \tag{2.16}$$

Equations (2.14)–(2.16) are summarized in Table I. As pointed out, for $n = -1$, Eq. (2.6) still agrees with (1.11), (1.12), (2.12), provided we choose P_{-1}, Q_{-1} as in Table I.

For general integer n, there results upon substituting (1.11), (1.12) in (2.8) the following recurrence equations for $P_n(u)$, $Q_n(u)$

$$(n + 1/2)Q_n(u) - uQ_n'(u)/2 = Q_{n-1}(u), \tag{2.17}$$

$$(n + 1/2)P_n(u) - uP_n'(u)/2 + u^2P_n(u) + uQ_n(u)/2 = P_{n-1}(1). \tag{2.18}$$

Equation (2.17) determines Q_n except for the term u^{2n+1}. Equation (2.18) then determines this term and P_n.

The relation (2.4) may be applied to express T_n as *power series* in x, by starting with the expansion for T_{-1} obtained from Eq. (2.6)

$$\begin{aligned} T_{-1} &= \frac{t^{-1/2}}{(\pi\rho cK)^{1/2}}\left[1 - u^2 + \frac{u^4}{2!} - \cdots\right] \\ &= \frac{1}{(\pi\rho cK)^{1/2}}\left[t^{-1/2} - \frac{x^2t^{-3/2}}{4k} + \frac{x^4t^{-5/2}}{2!(4k)^2} - \cdots\right] \end{aligned} \tag{2.19}$$

and integrating $(n + 1)$ times termwise with respect to t. A single integration yields

$$T_0 = \frac{1}{(\rho cKu)^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \cdot \frac{x^{2n}t^{-n+1/2}}{k^n 2^{2n}(n - 1/2)} + g_0(x), \tag{2.20}$$

where g_0 is the constant of integration which may depend on x. This may be determined by noting that $g_0(x)$ must satisfy Eq. (1.1), since T_0 and the series in (2.20) satisfy it. Hence $g_0(x)$ reduces to a first degree polynomial in x whose coefficients may be determined from the heat input condition at $x = 0$

$$-K \frac{\partial T}{\partial x}\bigg|_{x=0} = h(t), \tag{2.21}$$

and from

$$T_0(0, t) = \frac{2t^{1/2}}{(\rho c K\pi)^{1/2}} \tag{2.22}$$

which follows from (1.8) for $n = 0$. There results

$$T_0 = \frac{t^{1/2}}{(\rho c K\pi)^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}u^{2n}}{n!(n - 1/2)} - \frac{x}{K}. \tag{2.23}$$

Further t-integrations of (2.23) and similar determination of the constants of integration yield

$$T_1 = \frac{t^{3/2}}{(\rho c K\pi)^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^{n}u^{2n}}{n!(n - 1/2)(n - 3/2)} - \frac{1}{K}\left(xt + \frac{x^3}{3!k}\right) \tag{2.24}$$

$$T_2 = \frac{t^{5/2}}{(\rho c K\pi)^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}u^{2n}}{n!(n - 1/2)(n - 3/2)(n - 5/2)} - \frac{1}{K}\left(\frac{xt^2}{2!} + \frac{x^3 t}{3!k} + \frac{x^5}{5!k^2}\right). \tag{2.25}$$

Similarly, there results for T_n for any integer n, the termwise integrated series along with the polynomial

$$-\frac{1}{K}\left[\frac{xt^n}{n!} + \frac{x^3 t^{n-1}}{(n + 1)!k} + \cdots + \frac{x^{2n-1}}{(2n + 1)!k^n}\right]. \tag{2.26}$$

A further relation of interest between T_n

$$\frac{\partial^2 T_n(x, t)}{\partial x^2} = \frac{1}{k} T_{n-1}(x, t) \tag{2.27}$$

follows from (2.4) and the fact that T_n is a solution of (1.1).

The above expansions, while convergent for all u, converge slowly for large u, hence small t. For large u it is preferable to use asymptotic series of the form

$$\frac{T_n(x, t)}{t^{1/2}(\rho c K)^{1/2}} = \frac{2}{\pi^{1/2}} \exp(-u^2)\left[\frac{A_1}{u^{n+2}} + \frac{A_2}{u^{n+4}} + \cdots\right], \tag{2.28}$$

where $A_1, A_2, \cdots$ are constants. Indeed, for $n = -1$ such a series follows from (2.6) and

$$\operatorname{erfc}(v) = \frac{1}{\pi^{1/2}} \exp(-v^2)\left(\frac{1}{v} - \frac{1}{2v^3} + \frac{1.3}{2^2 v^5} - \cdots\right). \tag{2.29}$$

For $n = 1, 2, \cdots$ one assumes (2.28) and applies (2.4), (2.29) to determine the coefficients $A_1, A_2, \cdots$ for successive n.

Direct substitution of (1.11) in (1.1) shows that $f_n(u)$ is a solution of the differential equation

$$f_n''(u) + 2uf_n'(u) - (4n - 2)f_n(u) = 0 \tag{2.30}$$

which vanishes for $u = +\infty$.

3. Half integer and fractional power inputs. Operational expressions. By differentiating T_n with respect to x, one obtains solutions of (1.1) corresponding to the heat input (1.7) for values of n differing from integers by one-half. Indeed, consider the function*

$$T'_n = -\frac{\partial T_n(x, t)}{\partial x}, \tag{3.1}$$

where T_n with integer n are as in Secs. 1, 2. The heat input of T'_n at $x = 0$ is given by

$$h(t) = -K \left.\frac{\partial T'_n}{\partial x}\right|_{x=0} = K \left.\frac{\partial^2 T_n(x, t)}{\partial x^2}\right|_{x=0}. \tag{3.2}$$

Since T_n satisfies (1.1), $\partial^2 T_n/\partial x^2$ may be replaced by $(1/k)$ $(\partial T_n/\partial t)$, and hence, upon recalling (1.8),

$$h(t) = \frac{K}{k} \left.\frac{\partial T_n}{\partial t}\right|_{x=0} = \frac{\Gamma(1/2)t^{n-1/2}}{(\pi k)^{1/2}\Gamma(n + 1/2)}. \tag{3.3}$$

This proves the above statement regarding T'_n.

Recalling the form (1.11) for T_n, one obtains from (3.1)

$$T'_n = -(t^n/2K)f'_n(u), \qquad u = x/2(kt)^{1/2}, \tag{3.4}$$

and this can also be put in a form similar to (1.12).

Of special interest is the case $n = 0$ for which Eqs. (2.13), (3.4) yield

$$T'_0 = -\frac{f'_0(u)}{2K} = \frac{1}{K} \operatorname{erfc} \frac{x}{2(kt)^{1/2}}. \tag{3.5}$$

For $x = 0$, $t > 0$, this reduces to $1/K$. Hence, the function KT'_0 corresponds to a sudden temperature rise at $x = 0$, equal to 1. As shown in Fig. 3-1, at various instants the abscissas are changed in a fixed ratio. The heat input at $x = 0$ varies as $t^{-1/2}$.

It is of interest to note that KT'_0 can be obtained by dispensing with heat sources, but extending the medium to $x = -\infty$ and starting with the initial temperature

$$T(x, 0) = 2H(-t) = \begin{cases} 0 & \text{for} \quad x > 0, \\ 2 & \text{for} \quad x < 0, \end{cases} \tag{3.6}$$

(see Fig. 3-1 for the broken-line extensions).

From (2.27), (3.1) follows that the sequence of functions T'_n, T_n can be similarly extended to $x < 0$. In particular, the functions

$$\begin{aligned} u_0 &= KT'_0/2, \qquad u_1 = KT_0/2, \qquad u_2 = kKT'_1/2, \\ u_3 &= kKT_1/2, \qquad u_4 = k^2KT'_2/2, \qquad \cdots \end{aligned} \tag{3.7}$$

form a sequence of solutions of (1.1) satisfying the recurrence equations

$$\frac{\partial u_m}{\partial x} = -u_{m-1}, \qquad \frac{\partial u_m}{\partial t} = ku_{m-2} \tag{3.8}$$

and such that for $t = 0$

$$u_m(x, 0) = \begin{cases} 0 & \text{for} \quad x > 0, \\ (-x)^m/m! & \text{for} \quad x < 0. \end{cases} \tag{3.9}$$

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*The $-$ sign is chosen to render T'_n positive.

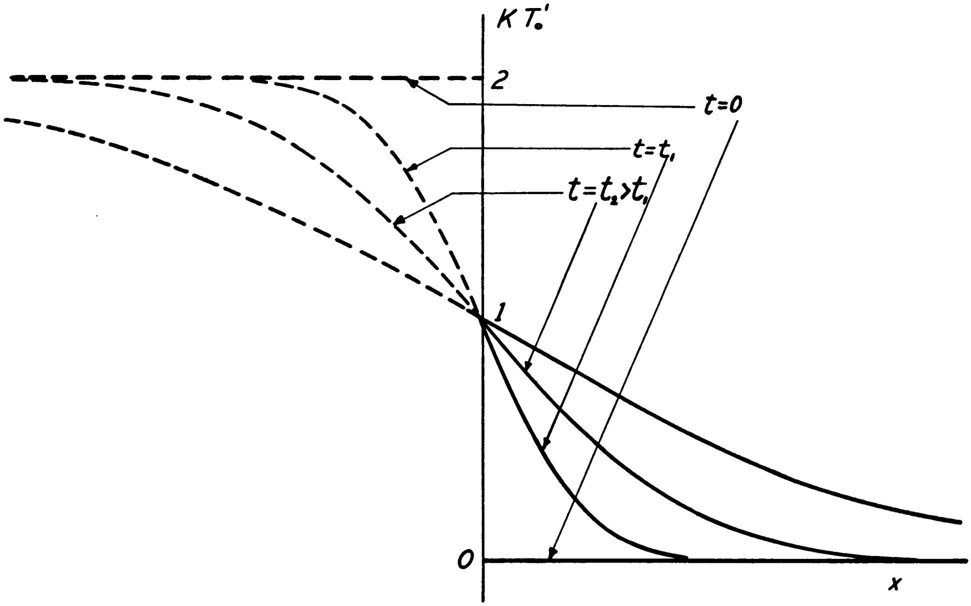

FIG. 3-1

This may be verified directly by considering (1.11), (1.12) for negative x, letting $t \to 0$, $u \to -\infty$, and noting that efrc $(-\infty) = 2$.

As indicated in (3.7), and noting (3.1),

$$\begin{aligned} u_{2n+1} &= k^n K T'_n/2, \\ u_{2n} &= -k^{n-1} K(\partial T_n/\partial x)/2. \end{aligned} \tag{3.10}$$

We may solve for u_m in terms of its initial values (3.9) by means of G from (1.2)

$$\begin{aligned} u_m(x, t) &= \int_{-\infty}^{+\infty} u_m(s, 0)G(x - s, t)\, ds \\ &= \int_{-\infty}^{0} s^m \exp\left[-(x - s)^2/4kt\right] ds/2(\pi kt)^{1/2} m!. \end{aligned} \tag{3.11}$$

Upon putting $s = x - v(4kt)^{1/2}$, there results

$$u_m(x, t) = \frac{2^{m-1}(kt)^{m/2}}{\pi^{1/2} m!}\, g_m(u), \tag{3.12}$$

where

$$u = x/(4kt)^{1/2}, \qquad g_m(u) = \int_u^\infty (v - u)^m \exp(-v^2)\, dv. \tag{3.13}$$

By expanding $(u - v)^m$ by the binomial theorem and integrating $v^k \exp(-v^2)\, dv$ by parts one can again express u_m in terms of $\exp(-u^2)$, erfc u, and polynomials in u.

With the possible exception of the application of the binomial theorem and Eq.

(3.10), Eqs. (3.8)-(3.13) apply equally well to all real $m > -1$, provided $m!$ be interpreted as $\Gamma(m+1)$.

For $x = 0$ there results (for both integer and non-integer m)

$$\begin{cases} u_m(0, t) = 2^{m-1}(kt)^{m/2}\pi^{-1/2}g_m(0)/m!, \\ \dfrac{\partial u_m(0, t)}{\partial x} = 2^{m-2}(kt)^{(m-1)/2}\pi^{-1/2}g_m'(0)/m!. \end{cases} \tag{3.14}$$

It is of interest to obtain an operational representation of the above solutions. To this end we replace the operator $\partial/\partial t$ by the symbol p in (1.1):

$$\frac{\partial^2 u}{\partial x^2} = \frac{p}{k}u. \tag{3.15}$$

Solving as if p were a constant, one obtains

$$u = B \exp[x(p/k)^{1/2}] + A \exp[-x(p/k)^{1/2}], \tag{3.16}$$

and dropping the first exponential for $x > 0$ (presumably because otherwise $u = \infty$ for $x = +\infty$),

$$u = A \exp[-(p/k)^{1/2}x]. \tag{3.17}$$

This yields for $x = 0$

$$u(0, t) = v(t) = A, \tag{3.18}$$

$$-K\frac{\partial u}{\partial x}\bigg|_{x=0} = h(t) = KA\left(\frac{p}{k}\right)^{1/2} = (K\rho c)^{1/2}(p)^{1/2}A. \tag{3.19}$$

Hence

$$h(t) = (K\rho c)^{1/2}p^{+1/2}v(t), \tag{3.20}$$

$$v(t) = (K\rho c)^{-1/2}p^{-1/2}h(t). \tag{3.21}$$

Interpreting p^{-n} as

$$p^{-n}h(t) = \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}h(s)\,ds, \tag{3.22}$$

for both integer and non-integer $n > 0$, there results

$$v(t) = T(0, t) = \frac{1}{(K\rho c)^{1/2}}\frac{1}{\Gamma(1/2)}\int_0^t \frac{h(s)}{(t-s)^{1/2}}\,ds \tag{3.23}$$

which agrees with Eq. (1.5).

Putting (3.20) in the form

$$h(t) = (K\rho c)^{1/2}p[p^{-1/2}v(t)], \tag{3.24}$$

there results

$$h(t) = \frac{1}{\Gamma(1/2)}(K\rho c)^{1/2}\frac{d}{dt}\int_0^t \frac{v(s)}{(t-s)^{1/2}}\,ds. \tag{3.25}$$

Equation (3.23) is an Abel integral equation for $h(t)$, and Eq. (3.25) yields its solution (see for instance, [1]).

For heat input given by

$$h(t) = t^n/\Gamma(n + 1) = p^{-n}1, \tag{3.26}$$

where n is either integer *or* fractional, Eqs. (3.17), (3.19) yield

$$T_n(0, t) = v(t) = \frac{1}{(K\rho c)^{1/2}} p^{-(n+1/2)}1 = \frac{1}{(K\rho c)^{1/2}} \frac{t^{n+1/2}}{\Gamma(n + 1/2)}, \tag{3.27}$$

while Eq. (3.17) yields

$$T_n = \frac{1}{(K\rho c)^{1/2}} \frac{1}{\Gamma(n + 1/2)} \frac{\exp[-(p/k)^{1/2}x]}{p^{n+1/2}} 1. \tag{3.28}$$

By interpreting these operational expressions as Brownwich integrals in accordance with

$$f(p)1 = \frac{1}{2\pi i} \int_L \frac{e^{pt}}{p} f(p)\, dp, \tag{3.29}$$

where L is a proper path of integration in the complex p-plane, one obtains an alternative (contour) integral representation for the solutions T_n and hence also for u_m.

Reference

1. E. T. Whittaker and G. N. Watson, "*A course of modern analysis*", 4th ed., Cambridge, 1958, p. 229