DUALITY IN QUADRATIC PROGRAMMING*†

BY

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Abstract. A proof, based on the duality theorem of linear programming, is given for a duality theorem for a class of quadratic programs. An illustrative application is made in the theory of elastic structures.

1. Introduction. Recent interest in quadratic programming has resulted in a series of computational methods for this type of problem. Some of these are described in [1, 2, 3, 4, 5]. Little emphasis, however, has been placed thus far on the concept of duality in quadratic programs. This concept, which has proved so valuable in linear programs, is investigated briefly in what follows.

In Sec. 5 a dual problem for a class of quadratic programming problems is formulated and the equality of the two objective forms is verified. Dennis [6] has indicated previously that such a duality existed based on the Kuhn-Tucker "equivalence theorem" [7]. The proof given here rests on the duality theorem for linear programs.

2. Notation. In what follows, matrix notation will be employed. Lower case letters, $x, y, \cdots$ will denote column vectors and capital letters $A, C, \cdots$ will represent matrices. Prime denotes transpose so that $x', y', \cdots$ are row vectors. The product $x'y$ is the inner product of the two vectors x and y.

A vector inequality will apply to each component of the vector, i.e., $x \geq 0$ indicates that each component of x is non-negative.

3. Duality in linear programming. The linear programming problem may be posed as follows. To minimize the linear form $p'x$ over all n-dimensional vectors x satisfying the constraints

$$Ax \geq b,$$
$$x \geq 0,$$

where p is an $n \times 1$ vector, b is an $m \times 1$ vector and A is an $m \times n$ matrix.

The dual problem to the above is to maximize $b'v$ over all m-dimensional vectors v satisfying

$$A'v \leq p,$$
$$v \geq 0.$$

The duality theorem [8, 9] states that if a solution to either problem exists and is finite, then a solution to the other problem also exists and indeed

$$\text{Minimum} \quad p'x = \text{Maximum} \quad b'v. \tag{3.1}$$

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4. A class of quadratic programs. A class of programs which has received considerable attention [3, 5, 6] is

$$\text{Minimize:} \quad f(x) = \tfrac{1}{2}x'Cx + p'x \tag{4.1}$$

subject to

$$Ax \geq b, \tag{4.2}$$

$$x \geq 0, \tag{4.3}$$

where C is a symmetric, positive semi-definite, $n \times n$ matrix and p, b, A and x are as in Sec. 3 above. This problem will be referred to as Problem I.

The symmetry restriction on C results in no loss of generality, while the positive semi-definiteness requirement assures that $f(x)$ is convex and that a local minimum is also a global one [3, 5].

In order to prove a duality theorem for this class of programs the following lemma is required.

Lemma. *If C is a symmetric, positive semi-definite matrix then for any vectors x and y*

$$y'Cy - x'Cx \geq 2x'C(y - x).$$

Proof. From positive semi-definiteness, for any x and y

$$(y - x)'C(y - x) \geq 0,$$

$$y'Cy \geq 2x'Cy - x'Cx.$$

Subtracting $x'Cx$ from both sides

$$y'Cy - x'Cx \geq 2x'C(y - x).$$

5. A duality theorem for quadratic programs. A dual problem to Problem I is

$$\text{Maximize:} \quad g(u, v) = -\tfrac{1}{2}u'Cu + b'v \tag{5.1}$$

subject to

$$A'v - Cu \leq p, \tag{5.2}$$

$$v \geq 0, \tag{5.3}$$

where u is an $n \times 1$ vector and v is an $m \times 1$ vector. This problem will be referred to as Problem II.

Theorem (*Dual*). (i) If $x = x_0$ is a solution to Problem I then a solution $(u, v) = (x_0, v_0)$ exists to Problem II. (ii) Conversely, if a solution $(u, v) = (u_0, v_0)$ to Problem II exists then a solution which satisfies $Cx = Cu_0$ to Problem I also exists. In either case

$$\text{Max} \quad g(u, v) = \text{Min} \quad f(x). \tag{5.4}$$

Proof. (A) Suppose first that $x = x_0$ is the minimizing solution of Problem I. Consider the following linear programming problem

$$\text{Minimize:} \quad F(x) = -\tfrac{1}{2}x_0'Cx_0 + x_0'Cx + p'x \tag{5.5}$$

subject to

$$Ax \geq b, \tag{5.6}$$

$$x \geq 0. \tag{5.7}$$

Denote this as Problem I′. Notice that the constraint sets for Problem I and Problem I′ are identical.

Now suppose there exists an x^* satisfying the constraints and such that

$$F(x^*) < F(x_0), \tag{5.8}$$

i.e.,

$$(x_0'C + p')(x^* - x_0) < 0.$$

It is easily verified that

$$x_1 = x_0 + k(x - x_0), \qquad 0 < k < 1$$

also satisfies the constraints. Now

$$f(x_1) - f(x_0) = k[(x_0'C + p')(x^* - x_0) + \tfrac{1}{2}k(x^* - x_0)'C(x^* - x_0)].$$

Choose k to be

$$k < -\frac{(x_0'C + p')(x^* - x_0)}{\frac{1}{2}(x^* - x_0)'C(x^* - x_0)}.$$

It follows that the term in square brackets is negative so

$$f(x_1) - f(x_0) < 0.$$

But $f(x_0) \leq f(x_1)$ since x_0 is the minimizing solution of Problem I so the inequality (5.8) cannot hold for any x, i.e., for all x

$$F(x) \geq F(x_0).$$

Thus x_0 minimizes $F(x)$ and is the optimal solution to Problem I′.

The dual problem to Problem I′ is (Sec. 3)

$$\text{Maximize:} \quad G(v) = -\tfrac{1}{2}x_0'Cx_0 + b'v \tag{5.9}$$

subject to

$$A'v \leq Cx_0 + p, \tag{5.10}$$

$$v \geq 0. \tag{5.11}$$

Denote this as Problem II′. By the duality theorem for linear programs, (3.1),

$$\text{Max} \quad G(v) = \text{Min} \quad F(x) = F(x_0).$$

If $v = v_0$ is a maximizing solution of Problem II′ then the last equation becomes

$$b'v_0 = x_0'Cx_0 + p'x_0 \, . \tag{5.12}$$

Consider now admissible solutions (u, v) to Problem II. In particular (x_0 , v_0) is admissible. Now from (5.1)

$$g(x_0 , v_0) - g(u, v) = -\tfrac{1}{2}x_0'Cx_0 + b'v_0 + \tfrac{1}{2}u'Cu - b'v$$

by the lemma

$$g(x_0 , v_0) - g(u, v) \geq x_0'C(u - x_0) + b'v_0 - b'v$$

and from (5.12)

$$g(x_0, v_0) - g(u, v) \geq x_0'Cu + p'x_0 - b'v. \tag{5.13}$$

Now from (5.2) and (4.3)

$$x_0'Cu \geq x_0'(A'v - p)$$

and from (4.2) and (5.3)

$$-b'v \geq -x_0'A'v.$$

Substituting these last two inequalities in (5.13)

$$g(x_0, v_0) - g(u, v) \geq x_0'A'v - x_0'p + p'x_0 - x_0'A'v = 0.$$

Thus (x_0, v_0) maximizes Problem II. Finally from (5.1), (5.12) and (4.1)

$$g(x_0, v_0) = -\tfrac{1}{2}x_0'Cx_0 + b'v_0 = \tfrac{1}{2}x_0'Cx_0 + p'x_0 = f(x_0) \tag{5.14}$$

which verifies the equality of the objective functions (4.1) and (5.1). This completes the proof of part (i) of the theorem.

(B) The converse will be proved by applying the above result to Problem II. Suppose a maximizing solution (u_0, v_0) of Problem II exists. Problem II may be rephrased

$$\text{Minimize:} \quad -g(u, v) = \tfrac{1}{2}u'Cu - b'v$$

subject to

$$Cu - A'v \geq -p,$$
$$v \geq 0.$$

Now let

$$u = r - s$$

where

$$r \geq 0,$$
$$s \geq 0.$$

Problem II then becomes

$$\text{Minimize:} \quad -G(r, s, v) = \tfrac{1}{2}(r, s, v)' \begin{pmatrix} C, & -C, & 0 \\ -C, & C, & 0 \\ 0, & 0, & 0 \end{pmatrix} \begin{pmatrix} r \\ s \\ v \end{pmatrix} + (0, 0, -b) \begin{pmatrix} r \\ s \\ v \end{pmatrix}$$

subject to

$$(C, -C, -A') \begin{pmatrix} r \\ s \\ v \end{pmatrix} \geq -p$$

$$r \geq 0$$
$$s \geq 0$$
$$v \geq 0.$$

This is now in the form of Problem I, and by part (i) of the theorem (already proved in A above) implies the existence of a solution to a dual problem which is

$$\text{Maximize:} \quad -\tfrac{1}{2}(w, y, z)' \begin{pmatrix} C, & -C, & 0 \\ -C, & C, & 0 \\ 0, & 0, & 0 \end{pmatrix} \begin{pmatrix} w \\ y \\ z \end{pmatrix} - p'x$$

subject to

$$Cx - Cw + Cy + 0z \leq 0 \tag{5.15}$$

$$-Cx + Cw - Cy + 0z \leq 0 \tag{5.16}$$

$$-Ax + 0w + 0y + 0z \leq -b \tag{5.17}$$

$$x \geq 0.$$

Moreover, the maximizing solution is required to satisfy

$$w - y = u_0 , \tag{5.18}$$

$$z = v_0 .$$

Inequalities (5.15) and (5.16) imply that

$$Cx = C(w - y) \tag{5.19}$$

so the dual problem may be rewritten

$$\text{Maximize:} \quad -\tfrac{1}{2}x'Cx - p'x = -f(x)$$

subject to

$$Ax \geq b,$$

$$x \geq 0,$$

which is exactly the original Problem I. From (5.18) and (5.19) then the optimizing solution x to Problem I must satisfy

$$Cx = Cu_0 .$$

Finally from Eq. (5.14) it follows that

$$\text{Min} \quad -g(u, v) = \text{Max} \quad -f(x),$$

which completes the proof of part (ii).

6. Computation of the dual variables. The proof in the preceding section also provides a means for calculating the dual variables (u_0 , v_0) once the primal variables, x_0 , have been found.

The vector u_0 is identical with x_0 . The vector v_0 is then a solution of a linear programming problem

$$\text{Maximize} \quad b'v$$

subject to

$$A'v \leq Cx_0 + p,$$

$$v \geq 0.$$

7. Other classes of problems. The quadratic problem (Problem I) may be formulated in various other ways with resulting changes in its dual (Problem II). Some of these, including the original, are tabulated below for reference.

Primal Problem	*Dual Problem*
Type I	
$\text{Min } \frac{1}{2} x'Cx + p'x$ $Ax \geq b$ $x \geq 0$	$\text{Max } - \frac{1}{2} u'Cu + b'v$ $A'v - Cu \leq p$ $v \geq 0$
Type II	
$\text{Min } \frac{1}{2} x'Cx + p'x$ $Ax \geq b$	$\text{Max } - \frac{1}{2} u'Cu + b'v$ $A'v - Cu = p$ $v \geq 0$
Type III	
$\text{Min } \frac{1}{2} x'Cx + p'x$ $Ax = b$ $x \geq 0$	$\text{Max } - \frac{1}{2} u'Cu + b'v$ $A'v - Cu \leq p$
Type IV	
$\text{Min } \frac{1}{2} x'Cx + p'x$ $Ax = b$	$\text{Max } - \frac{1}{2} u'Cu + b'v$ $A'v - Cu = p$

The Type IV problem may, of course, be treated by standard Lagrange multiplier techniques. The dual problem for a problem of this type has been given previously [10]. Indeed, v are the multipliers for the original problem.

Notice that at the optimum, in all types listed above,

$$u = x. \tag{7.1}$$

8. An application to elasticity. As an application of the duality theorem for quadratic programs, consider the problem of determining the elastic solution of a plane pin-jointed truss consisting of n bars and m joints ($n \geq 2m - 3$). The truss is externally statically determinate and the applied loads lie in the plane of the truss.

The problem may be formulated as minimizing the strain energy subject to equilibrium constraints (Castigliano's Second Theorem).

If S_j denotes the force in the jth member and A_j, E_j, L_j are its cross-sectional area, elastic modulus and length respectively, then the strain energy U is

$$U = \frac{1}{2} \sum_{j=1}^{n} \frac{L_j}{A_j E_j} S_j^2$$

and the equilibrium conditions may be written [11]

$$\sum_{j=1}^{n} a_{ij} S_j = F_i \qquad (i = 1, 2, \cdots, 2m - 3),$$

where F_i is the force component at the ith joint. The a_{ij} depend on the geometrical configuration and are essentially direction cosines of the angles between the bars and the coordinate axes.

This is a problem of Type IV and the dual problem to this minimum problem is from Sec. 7

$$\text{Maximize:} \quad -\tfrac{1}{2} \sum_{j=1}^{n} \frac{L_j}{A_j E_j} S_j^2 + \sum_{i=1}^{2m-3} F_i u_i$$

subject to

$$\sum_{i=1}^{2m-3} a_{ij} u_i - \frac{L_j}{A_j E_j} S_j = 0 \qquad (j = 1, 2, \cdots, n).$$

The use of the variables S_j in the dual is justified by Eq. (7.1). Making use of Hooke's law which gives the elongation, e_j, of the jth bar as

$$e_j = \frac{L_j}{A_j E_j} S_j ,$$

the problem becomes

$$\text{Maximize:} \quad -\tfrac{1}{2} \sum_{j=1}^{n} \frac{A_j E_j}{L_j} e_j^2 + \sum_{i=1}^{2m-3} F_i u_i$$

subject to

$$\sum_{i=1}^{2m-3} a_{ij} u_i - e_j = 0 \qquad (j = 1, 2, \cdots, n).$$

If, for the moment it is assumed that the u_i are the displacement components of the ith joint, the objective function for the dual problem is the negative of the total potential energy and the constraints become the compatibility equations. The dual problem is, therefore, equivalent to the Theorem of Minimum Potential Energy. The equality of the objective functions results in a restatement of the Principle of Virtual Work.

Other applications to electrical networks containing resistors, diodes and voltage and current sources may be found in [6].

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