QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME XVIII JULY · 1960 NUMBER 2

QUARTERLY OF APPLIED MATHEMATICS

This periodical is published quarterly by Brown University, Providence 12, R. I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:

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Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

BOOK REVIEWS

Introduction to functional analysis. By Angus E. Taylor. John Wiley & Sons, Inc., New York, Chapman & Hall, Ltd., London, 1958. xvi + 423 pp. $12.50.

The theory of linear operators and topological vector spaces grew from the study of the kind of linear differential equations and integral equations which occur in applied mathematics. Now that the pure mathematician has developed these theories, the applied mathematician is feeling the need to learn something of them. The publication four years ago of Bernard Friedman's "Principles and Techniques of Applied Mathematics" clearly indicated the power of the new abstract methods in the analysis of practical problems. The book by Professor Taylor, under review, is the first *textbook* on functional analysis to appear in English (only the first part of the translation of the book by Kolmogorov and Fomin has so far appeared), and it is so well written that it will almost certainly become the standard text on the subject.

The first two chapters "The Abstract Approach to Linear Problems" and "Topologies" provide the basis for the long third chapter on "Topological Linear Spaces" which, despite its general title, is almost entirely confined to a discussion of normed spaces. Chapter 4, "General Theorems on Linear Operators", is concerned with the study of continuous linear operators in normed linear spaces and contains basic results such as the "closed-graph theorem", the principle of uniform boundedness, and the "projection theorem". The next two chapters," Spectral Analysis of Linear Operators" and "Spectral Analysis in Linear Space", form, for the applied mathematician, the core of the book; the former deals with spectral theory in general Banach spaces while the latter is concerned with bounded operators in Hilbert space. The book ends with a chapter on integration and linear functionals.

The whole book is exceedingly well written. In his preface, Professor Taylor writes: "This book is an introduction, not a treatise. It is meant to open doors for the student and to give him understanding and preparation which will help him to push on, if he wishes, to the new frontiers of modern mathematics, carrying with him a clearer realization of the structure of classical mathematics". This aim he has fully accomplished. New ideas are introduced carefully (and never without motivation) and many illustrations of the applicability of the general theory are given. There are also problems for the reader to attempt.

This is the most suitable book available for a graduate course on functional analysis and for independent reading. An applied mathematician with little previous experience of abstract mathematics might find it difficult going. He had better begin by reading the book by Friedman or (if he can read Russian) Vulikh's *"Vedenie v Funktshional'nii Analiz"*. But for the man who is well equipped and is prepared to make the effort, Taylor's book is the most rewarding yet on functional analysis, though some pure mathematicians might think that already it is a little "old-fashioned".

I. N. Sneddon

Handbook of automation, computation, and control. Edited by Eugene M. Grabbe, Simon Ramo, Dean E. Wooldridge. John Wiley & Sons, New York, Chapman & Hall, Ltd., London, 1959. xxiii + 1096 pp. $17.50.

The second volume of this unique handbook treats of: A. Computer Terminology, B. Digital Computer Programming, C. The Use of Digital Computers and Data Processors, D. Design of Digital Computers, E. Design and Application of analog Computers, F. Unusual Computer Systems.

The applied mathematician will mainly be interested in Section B, in which John W. Carr III gives, in 270 pages, a masterly survey of programming and coding. About 150 pages are devoted to advanced topics, here presented for the first time in easily accessible form, principally the construction and use of automatic programming systems with examples drawn from existing U. S. and Soviet compilers, interpreters and translators. The instruction logic of some common scientific and business computers, both large and small scale, is also given. The emphasis on existing computers and programming systems instead of hypothetical ones, often presented in books, is particularly welcome.

Walter F. Freiberger

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BOOK REVIEWS

(*Continued from p. 122*)

Einführung in Theorie und Anwendung der Laplace-Transformation. By Gustav Doetsch. Birkhäuser Verlag, Basel and Stuttgart, 1958. 301 pp. $9.15.

Professor Doetsch's three volume *Handbuch der Laplace-Transformation* has established itself in the last ten years as the standard treatise on the Laplace transformation and its applications. The volume under review aims at being a short introduction to the subject. The book is divided into 28 sections with chapter divisions. In the first eleven sections the elementary properties of the Laplace transform are developed. Sections 12–15 describe the application to ordinary differential equations with constant coefficients, systems of ordinary differential equations and difference equations, while §16 is concerned with the behaviour of the Laplace transform at infinity. The central part of the book (§§17–25) is devoted to a discussion of the inversion formula and related topics including Parseval's theorem and the asymptotic behaviour of the image function and the original function. The final three sections are concerned with applications: §26 considers the solution of ordinary differential equations with polynomial coefficients (in particular, Bessel's equation), §27 deals with partial differential equations in two independent variables (the diffusion equation and the telegraphy equation), and the last section is given over to a discussion of the solution of integral equations of the "Faltung" type.

The book is clearly and attractively written and anyone wishing a sound introduction to the theory of the Laplace transform is well advised to read it. For all that, an applied mathematician or engineer beginning the subject would probably find Churchill's "Operational Mathematics" more to his taste.

I. N. SNEDDON

Theory of differential equations. By A. R. Forsyth. Dover Publications, Inc., New York, 1960. Vol. 1, xiii + 340 pp.; Vol. 2, xi + 344 pp.; Vol. 3, x + 391 pp.; Vol. 4, xvi + 534 pp.; Vol. 5, xx + 478 pp.; Vol. 6, xiii + 596 pp. $15.00.

This unabridged republication of the well-known work consists of three volumes each of which combines two volumes of the original edition.

Theoretical elasticity. By Carl E. Pearson. Harvard University Press, Cambridge, 1959. 218 pp. $6.00.

This book evidently grew out of the author's lectures given at Harvard University, the purpose of its publication being ". . . in part to discuss modern methods of elasticity theory in a form which does not require extensive mathematical experience and in part to provide a compact and convenient summary of such methods." The selection of topics discussed deviates from the usual coverage in books on elasticity. Thus, for example, plane problems and Saint-Venant's torsion problem are excluded. On the other hand, a few of the topics dealt with in the latter part of the book (notably parts of chapters 6, 7, and 10), may not be found in other books dealing exclusively with elasticity. No attempt is made to supply adequate and appropriate references to the brief treatment of the various topics; a few references are cited as footnotes. The author is unusually consistent in his omission of references, however, since for example, a good deal of the content of the last chapter is based on the author's own excellent paper [*Quart. Appl. Math. 14*, 133-144, 1957] to which he does not refer.

Chapters 1 and 2 (45 pages) are concerned with a review of the vector algebra, vector calculus, index notation, and Cartesian tensors. Chapters 3, 4, 5 (63 pages) are respectively devoted to the analysis of stress, analysis of strain, and derivation of the basic equations together with some of the fundamental theorems in the linear theory of elasticity. The next two chapters (6 and 7) are entitled "General Solutions" and "Variational Methods," respectively. The former, following a brief review of potential theory, contains an account of various integral representations (such as Betti's formula for dilitation)

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BOOK REVIEWS

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and other representations of the general solution of the displacement differential equations of equilibrium (e.g., Boussinesq-Papkovich and Galerkin), as well as a clear exposition of the basic concepts of function space methods in elasticity. Chapter 7 deals with variational methods, such as the Rayleigh-Ritz procedure, the method of Lagrangian multiplier and application of the function-space methods.

The thermodynamic approach adopted in Chapter 8 (entitled "Thermoelasticity"), essentially that appropriate to a simple homogeneous substance, falls short of recent developments on the subject [see, e.g., M. A. Biot, *J. Appl. Phys.*, *27*, 240-253 (1956) and M. Lessen, *Quart. Appl. Math. 15*, 105-108 (1957)]. Furthermore, the treatment of the subject as given in Chapter 8 is quite elementary and may be found elsewhere [e.g., at various places in Zemansky's *Heat and Thermodynamics*, McGraw-Hill, 1957, and Searle's *Experimental Elasticity*, Cambridge, 1908, pp. 20-29]; the statement on p. 159 that Gibbs entropy relation "... holds for all processes, reversible or not" is misleading, as the proof of its general validity for irreversible processes has not been established. The remaining two chapter-headings are Time-Dependent Problems and Nonlinear Elasticity. Chapter 9 is a cursory treatment of vibrations of and wave propagations in isotropic elastic bodies. Finally, Chapter 10, following a short account of nonlinear elasticity (about 7 pages), deals with the problem of elastic stability.

On the whole the book is readable and provides a useful summary of some topics in elasticity. Despite its title, however, this book is not on a par with *Theoretical Elasticity* by Green and Zerna, or Sokolnikoff's *Mathematical Theory of Elasticity*.

P. M. NAGHDI

Celestial mechanics. By E. Finlay-Freundlich. Pergamon Press, New York, 1958. viii + 150 pp. $7.50.

Celestial mechanics is a very old branch of applied mathematics and one would suppose that very little new can be said about it. This is true as far as fundamentals are concerned. But interest in the subject has been renewed in recent years partly because of the "space age" concern with rockets and artificial satellites and even more because of the astronomer's endeavor to learn more of the nature of binary stellar systems. Professor Finlay-Freundlich, who has just retired from his professorship in St. Andrews University in Scotland, has here provided a new and attractive brief summary of the basic principles of celestial mechanics with some emphasis on current applications.

The book contains a very clear statement of the n-body problem and the reason why it is unsolvable in the general sense (Bruns-Poincaré theorem). This is followed by chapters on the application of the Hamilton-Jacobi theory (transformation theory of mechancis) to the three-body problem, and the theory of perturbations. The two-body problem for extended deformable bodies constitutes the really modern material in the book, and there is a good discussion of close binary systems with tidal deformations. Pedagogically speaking the weakest part of the book is the chapter on the application of relativistic mechanics to the two body problem. Here the relativistic four-dimensional line element is introduced without explanation and the ensuing derivation of the Einstein formula for the advancement of the perihelion of Mercury assumes the form merely of a mathematical exercise.

On the whole the author is to be congratulated on the clarity of his treatment and the large amount of fundamental material he has compressed into so small a space.

R. B. LINDSAY

Testing statistical hypotheses. By E. L. Lehmann. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1959. xiii + 369 pp. $11.00.

This volume is based, in part, on a well-known set of lecture notes of the author. These notes also included a discussion of the theory of estimation. The book is limited to a discussion of testing problems. This may be due to limitations of space and perhaps also a belief that the relevant theory in this domain is in a more coherent form.

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BOOK REVIEWS

(*Continued from p. 154*)

The first chapter discusses the "general decision problem" a viewpoint that appears to have become canonical in most statistical discussions today. Various notions of optimum procedures, Bayes and minimax procedures as well as the concepts of invariance and unbiasedness are introduced. Chapter 2 sketches out the appropriate background in probability theory required for the book. The notion of sufficiency is discussed in a measure theoretic background. From this point on the following chapters discuss and elaborate in some detail the approaches that have been indicated in the first chapter. Uniformly most powerful tests are discussed in chapter 3. The Neyman-Pearson lemma and extensions thereof are introduced. A variety of contexts in which uniformly most powerful tests exist are noted. Unbiased tests are examined in the next two chapters. Invariance and a discussion of linear hypotheses follow in chapters 6 and 7 respectively. The minimax principle is examined in chapter 8 and the book concludes with an appendix.

The book should be of considerable interest to those interested in a formal and detailed exposition of theory of testing and the criteria that have been introduced in this domain. The author is exceedingly well qualified since much of his research has been intensively directed toward the theory of testing. The orientation of the book is such that it presents techniques directed towards proving the optimality or appropriateness of tests proposed some time ago (perhaps partly on intuitive grounds) rather than as tools for work in new areas.

M. Rosenblatt

Statistical independence in probability, analysis and number theory. By Mark Kac. Published by The Mathematical Association of America, distributed by John Wiley & Sons, Inc., New York, 1959. xiv + 93 pp. $3.00.

This little volume is based on the Hedrick lectures given by the author during the summer of 1955. The reader is assumed to have some familiarity with the Lebesgue integral and elements of Fourier analysis and number theory. The object of the author is to show how probabilistic ideas, in particular that of independence, arise in concrete problems that one would ordinarily think of as outside the range of probability theory. The first chapter introduces the Rademacher functions as an example of independent functions as well as the usual statistical model of a fair coin-tossing game. In chapter two, Borel's law on the normality of real numbers and convergence (or divergence) of series with random signs are considered. The central limit theorem is briefly discussed in chapter 3. Probabilistic ideas are applied to some number theoretic problems in chapter 4, which is called "Primes play a game of change." The concluding chapter discusses the ergodic theorem as suggested by kinetic theory and then applies the ergodic theorem to a problem on continued fractions. The author succeeds admirably since the book gives the reader a stimulating contact with the ideas of probability theory and their power in investigating problems in other fields. There is a strong emphasis on the analytic aspect of probability theory rather than measure theoretics.

M. Rosenblatt

Confluent hypergeometric functions. By L. J. Slater. Cambridge University Press, New York, 1960. ix + 247 pp. $12.50.

This book contains a detailed and careful discussion of all properties of the confluent hypergeometric functions, including Kummer and Whittaker functions, which a mathematical physicist is likely to require. The author describes the basic differential equations satisfied by the functions, derives the latter's differential and integral properties and outlines the underlying functional analysis necessary in the development of the asymptotic expansions. She presents related differential equations and special cases of the functions, such as the Coulomb and Schrödinger wave equations, and concludes with a table of 120 pages, calculated on the EDSAC I, of the function ${}_1F_1[a; b; x]$ over those ranges most likely to be useful to the computer. There is an extensive bibliography.

Walter Freiberger