THE STOKES FLOW ABOUT A SPINDLE*

BY

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I. Statement of the problem for an axially symmetric body. The Stokes flow of a viscous, incompressible fluid about a body is defined by the assumption that inertial effects are negligible in comparison with those of viscosity, or, more precisely, that the Reynolds number of the flow is small. In the case in which the flow is two-dimensional or has radial symmetry, the introduction of a stream function and the Stokes assumption together with the no-slip boundary condition on the body and an appropriate assumption at infinity, yields a boundary value problem which has been solved in a number of instances. See Dryden, Murnaghan and Bateman [1], pp. 295-312 and the references of [2].

In the axi-symmetric case a body (or configuration of bodies) having an axis of symmetry is immersed in a flow which has a uniform velocity parallel to the axis of symmetry. It is then reasonable to suppose that the resulting flow pattern is identical in all planes through the axis of symmetry. If we introduce cylindrical coordinates (x, r, θ) into the flow space, where $x(-\infty \leq x \leq \infty)$ is measured along the axis of symmetry, $r(0 \leq r \leq \infty)$ normal to this axis, and $\theta(0 \leq \theta < 2\pi)$ is measured with respect to an arbitrary plane through the axis of symmetry, then θ plays no further role in our analysis because of the axial symmetry. Now let the velocity of the fluid be $\mathbf{u}(x, r) = (u_x(x, r), u_r(x, r))$ and introduce a stream function $\psi(x, r)$ by the equations

$$u_x = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{1.1}$$

By a well-known procedure (Payne and Pell [2]; Milne-Thomson [3], pp. 521-523) the differential equation to be satisfied in the region of flow D is found to be

$$L_{-1}^2 \psi = 0, \tag{1.2}$$

where

$$L_{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{1.3}$$

Let the trace of the boundary of the body in a meridional plane be C (Fig. 1). Then the condition of vanishing velocity on C can be stated in the form

$$\left.\begin{aligned} \psi &= 0 \qquad (1.4) \\ \frac{\partial \psi}{\partial n} &= 0 \qquad (1.5) \end{aligned}\right\} \quad \text{on} \quad C,$$

where $\mathbf{n}$ is the unit normal to C exterior to the body.

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If the uniform velocity of the flow at infinity is $\mathbf{u} = (U, 0)$, then ψ must satisfy the condition

$$\lim_{\rho\to\infty} \psi = \tfrac{1}{2}r^2U + O(\rho), \tag{1.6}$$

where

$$\rho^2 = x^2 + r^2. \tag{1.7}$$

II. Representation of the solution. It is expedient to define a second stream function ψ_1 by the relation

$$\psi = \tfrac{1}{2}Ur^2 - \psi_1 . \tag{2.1}$$

We then find that ψ_1 must satisfy the equation

$$L_{-1}^2\psi_1 = 0 \tag{2.2}$$

in D, subject to the conditions

$$\psi_1 = \tfrac{1}{2}Ur^2, \tag{2.3}$$

$$\frac{\partial\psi_1}{\partial n} = Ur\frac{\partial r}{\partial n}, \tag{2.4}$$

on C, as well as the condition that ψ_1 give rise to a vanishing velocity at infinity.

Following Weinstein [4], solutions of

$$L_k(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{k}{r}\frac{\partial v}{\partial r} = 0 \tag{2.5}$$

k real, are known as generalized axially symmetric potentials, and denoted by ψ^k. Payne [5] has shown that in certain regions any solutions of the repeated operator equation (2.2) can be represented as a linear combination of any two of the functions

a) $r^2\psi^3$, b) $xr^2\psi^3$, c) $\rho^2r^2\psi^3$,

d) $r^2\psi^1$, e) $r^4\psi^5$.

In a previous paper [2] the authors have used this theorem to obtain the solution of certain problems in Stokes flow; it will be used here to obtain still another.

III. The flow about a spindle. We introduce bipolar coordinates (ξ, η) into the (x, r) plane by the transformation

$$z = ib\cot(\zeta/2), \tag{3.1}$$

where $z = x + ir$, $\zeta = \xi + i\eta$, and $b > 0$ is a constant. In terms of the (ξ, η) coordinates

$$x = \frac{b\sinh\eta}{\cosh\eta - \cos\xi}, \qquad r = \frac{b\sin\xi}{\cosh\eta - \cos\xi}, \tag{3.2}$$

where $-\infty < \eta < \infty$ and $0 \le \xi \le \pi$. We define a spindle to be an object whose surface is obtained by revolving the curve $\xi = \xi_0 (0 < \xi_0 < \pi)$ about the x axis. This curve is the arc lying in $r \ge 0$ of the circle which passes through $(\pm b, 0)$ and has its center at $(0, -b\cot\xi_0)$. The surface of the spindle is thus $\xi = \xi_0$ and the region of flow D is $0 \le \xi \le \xi_0$, $-\infty < \eta < \infty$ (Fig. 1). It is advantageous in considering flow about the spindle to choose ξ and η as independent variables.

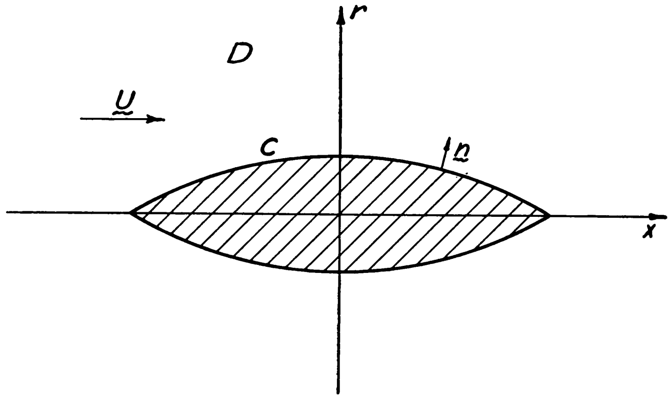

FIG. 1.

Using the results of [5] we represent the solution of (2.2) in the form

$$\psi_1 = r^2\psi^1 + (\rho^2 - b^2)r^2\psi^3. \tag{3.3}$$

It is not difficult to see [6] that any $\psi^{2n+1}(\xi, \eta)$ which is even in η may be expressed in the form

$$\psi^{2n+1} = (\cosh\eta - \cos\xi)^{(2n+1)/2} \int_0^\infty F(\alpha)K_\alpha^{(n)}(\cos\xi)\cos\alpha\eta\, d\alpha, \tag{3.4}$$

where $F(\alpha)$ is an arbitrary function of α,

$$K_\alpha(\cos\xi) = P_{i\alpha-1/2}(\cos\xi) \tag{3.5}$$

(known as the conal function; see Hobson [7], pp. 444-453) and $K_\alpha^{(n)}(\lambda) \equiv d^nK_\alpha(\lambda)/d\lambda^n$. Substitution of (3.4) in (3.3) permits us to write ψ_1 in the form

$$\psi_1 = \tfrac{1}{2}Ur^2(s - t)^{1/2} \int_0^\infty [A(\alpha)tK_\alpha^{(1)}(t) + B(\alpha)K_\alpha(t)]\cos\alpha\eta\, d\alpha, \tag{3.6}$$

where

$$s = \cosh\eta \qquad t = \cos\xi \tag{3.7}$$

and $A(\alpha)$, $B(\alpha)$ are functions to be determined in such a way that the boundary conditions (2.3-.4) are satisfied. The first of these yields

$$\int_0^\infty [A(\alpha)t_0K_\alpha^{(1)}(t_0) + B(\alpha)K_\alpha(t_0)]\cos\alpha\eta\, d\alpha = (s - t_0)^{-1/2}. \tag{3.8}$$

Next we observe that $\partial(\ )/\partial n = 0$ is equivalent to $\partial(\ )/\partial\xi = 0$ on $\xi = \xi_0$, and make use of (3.8) to show that (2.4) can be written in the form

$$\int_0^\infty \left\{A(\alpha)\frac{\partial}{\partial t_0}[t_0K_\alpha^{(1)}(t_0)] + B(\alpha)\frac{\partial K_\alpha(t_0)}{\partial t_0}\right\}\cos\alpha\eta\, d\alpha = \frac{\partial}{\partial t_0}(s - t_0)^{-1/2}. \tag{3.9}$$

The conal function $K_\alpha(-t)$ may be defined ([7], p. 446) by

$$K_\alpha(-t) = \frac{2^{1/2}}{\pi}\cosh\alpha\pi \int_0^\infty [\cosh u - \cos\xi]^{-1/2}\cos\alpha u\, du. \tag{3.10}$$

But for $0 < \xi_0 < \pi$, $(\cosh \eta - \cos \xi_0)^{-1/2}$ satisfies the hypotheses of the Fourier integral theorem, and utilizing (3.10) we thus have

$$
\begin{aligned}
(s - t_0)^{-1/2} &= \frac{2}{\pi} \int_0^\infty \cos \alpha\eta \left\{ \int_0^\infty [\cosh u - t_0]^{-1/2} \cos \alpha u \, du \right\} d\alpha \\
&= 2^{1/2} \int_0^\infty \frac{K_\alpha(-t_0) \cos \alpha\eta}{\cosh \alpha\pi} \, d\alpha,
\end{aligned} \tag{3.11}
$$

where $t_0 = \cos \xi_0$. Thus the right hand member of (3.8) may be replaced by the integral (3.11). For $0 < \xi_0 < \pi$ one may differentiate (3.11) with respect to t_0, and obtain a representation for the right-hand member of (3.9). We are led in this way to two linear equations for A and B whose solution is

$$A(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi} [K_\alpha(-t_0)K_\alpha^{(1)}(t_0) - K_\alpha^{(1)}(-t_0)K_\alpha(t_0)], \tag{3.12}$$

$$B(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi} \left[t_0 K_\alpha^{(1)}(t_0)K_\alpha^{(1)}(-t_0) - K_\alpha(-t_0) \frac{d}{dt_0} (t_0 K_\alpha^{(1)}(t_0))\right], \tag{3.13}$$

where

$$\Omega = t_0[K_\alpha^{(1)}(t_0)]^2 - K_\alpha(t_0) \frac{d}{dt_0} (t_0 K_\alpha^{(1)}(t_0)). \tag{3.14}$$

More compact expressions for A and B can be obtained, however. Suitable regrouping of the terms of Ω and repeated use of the fact that K_α satisfies a Legendre equation yields

$$\frac{d}{dt} [(1 - t^2)\Omega] = -2K_\alpha(t)K_\alpha^{(2)}(t). \tag{3.15}$$

Integration of this with respect to t and division of the result by $1 - t^2$ gives

$$\Omega = \frac{2}{1 - t_0^2} \int_{t_0}^1 K_\alpha(\tau)K_\alpha^{(2)}(\tau) \, d\tau, \qquad -1 < t_0 < 1. \tag{3.16}$$

The bracketed quantity in A is simply the Wronskian of $K_\alpha(-t)$ and $K_\alpha(t)$ evaluated at t_0, which (Neumann [8], pp. 207-210) is $-2 \cosh \alpha\pi/\pi(1 - t_0^2)$. Thus we obtain from (3.12)

$$A(\alpha) = -\frac{2^{1/2}}{\pi} \left(\int_{t_0}^1 K_\alpha(\tau)K_\alpha^{(2)}(\tau) \, d\tau \right)^{-1} \tag{3.17}$$

for $-1 < t_0 < 1$.

By essentially the same procedure as was used in discussing Ω, it can be shown that

$$\frac{d}{dt} \left\{ (1 - t^2)\left[tK_\alpha^{(1)}(t)K_\alpha^{(1)}(-t) - K_\alpha(-t) \frac{d}{dt} (tK_\alpha^{(1)}(t))\right] \right\} = -2K_\alpha^{(2)}(t)K_\alpha(-t) \tag{3.18}$$

holds for $-1 < t < 1$. This may be integrated with respect to t from t_0 $(-1 < t_0 \leq 1)$ to 1 and divided by $1 - t_0^2$ in order to obtain the bracketed portion of $B(\alpha)$. It will be noted that (3.18) is not valid at $t = \pm 1$, since these are singular points for either $K_\alpha(t)$ or $K_\alpha(-t)$ and their derivatives, but the relations ([8], pp. 207 and 209)

$$\lim_{t\to 1} (1 - t)^j K_\alpha^{(j)}(-t) = \frac{\Gamma(j)}{\pi} \cosh \alpha\pi, \qquad j = 1, 2, \cdots \tag{3.19}$$

$$K_\alpha(1) = 1 \tag{3.20}$$

$$K_\alpha^{(j)}(1) = \frac{1}{2^j \Gamma(j+1)} \left[\alpha^2 + \left(\frac{1}{2}\right)^2\right]\left[\alpha^2 + \left(\frac{3}{2}\right)^2\right] \cdots \left[\alpha^2 + \left(\frac{j-1}{2}\right)^2\right], \qquad j = 1, 2, \cdots \tag{3.21}$$

permit the integration just mentioned to be carried out up to $t = 1$. We find then that (3.13) yields

$$B(\alpha) = A(\alpha)\left\{\frac{1}{2}\left(\alpha^2 + \frac{1}{4}\right) - \frac{\pi}{\cosh \alpha\pi} \int_{t_0}^{1} K(-\tau) K^{(2)}(\tau)\, d\tau\right\} \tag{3.22}$$

for $-1 < t_0 < 1$.

The insertion of (3.17) and (3.22) in (3.6), and of the result in (2.1) gives the stream function for the Stokes flow about the spindle.

It is easily verified that for $\xi_0 = \pi/2 (t_0 = 0)$, Eq. (3.6) with $A(\alpha)$ and $B(\alpha)$ defined by (3.12) and (3.13) [or by (3.17) and (3.22)] gives the correct result for the flow about a sphere. In this case we note that

$$\lim_{t_0\to 0} K_\alpha(t_0) = \lim_{t_0\to 0} K_\alpha(-t_0) \tag{3.23}$$

and

$$\lim_{t_0\to 0} K^{(1)}(t_0) = -\lim_{t_0\to 0} K^{(1)}(-t_0). \tag{3.24}$$

For $t_0 = 0$, (3.12) and (3.13) then gives

$$A(\alpha) = -2^{3/2}/\cosh \alpha\pi, \tag{3.25}$$

$$B(\alpha) = 2^{1/2}/\cosh \alpha\pi. \tag{3.26}$$

From (3.6), the expression for ψ_1 becomes

$$\psi_1 = 2^{1/2} U r^2 (s - t)^{1/2} \int_0^\infty \frac{[\frac{1}{2} K_\alpha(t) - t K_\alpha^{(1)}(t)]}{\cosh \alpha\pi} \cos \alpha\eta\, d\alpha. \tag{3.27}$$

The integral on the right hand side of (3.27) is easily evaluated from (3.11) and the expression obtained by differentiation of (3.11) with respect to t_0. We merely replace ξ_0 by $\pi - \xi$ in the resulting integral formulas and obtain at once the following expression for ψ_1:

$$\psi_1 = \tfrac{1}{2} U r^2 (s - t)^{1/2} \{(s + t)^{-1/2} + t(s + t)^{-3/2}\}. \tag{3.28}$$

This may be rewritten as

$$= \tfrac{1}{4} U r^2 (s - t)^{1/2} \{3(s + t)^{-1/2} - (s - t)(s + t)^{-3/2}\}$$

$$= \tfrac{1}{4} U r^2 \{3b\rho^{-1} - b^3 \rho^{-3}\} \tag{3.29}$$

where b is the radius of the sphere. This is the well known result of Stokes (see [3]).

IV. Drag of the spindle. It was shown in [2] that the drag P of an axially symmetric body for which the flow region D is simply connected is given by

$$\frac{P}{8\pi\mu} = \lim_{\rho\to\infty} \frac{\rho\psi_1}{r^2}. \tag{4.1}$$

Substituting from (1.7), (3.2), and (3.6) in (4.1) we obtain

$$\begin{aligned}\frac{P}{8\pi\mu} &= \frac{Ub}{2} \lim_{\substack{\xi\to 0\\ \eta\to 0}} (s + t)^{1/2} \int_0^\infty [A(\alpha)tK_\alpha^{(1)}(t) + B(\alpha)K_\alpha(t)] \cos\alpha\eta \, d\alpha, \\ &= \frac{2^{1/2}}{2} Ub \int_0^\infty [A(\alpha)K_\alpha^{(1)}(1) + B(\alpha)K_\alpha(1)] \, d\alpha.\end{aligned}$$

From (3.20-.21) we obtain $K_\alpha(1)$ and $K_\alpha^{(1)}(1)$, and thus the drag of the spindle becomes

$$P = 2^{3/2}\pi\mu bU \int_0^\infty [2B(\alpha) - (\alpha^2 + \tfrac{1}{4})A(\alpha)] \, d\alpha \tag{4.2}$$

or

$$P = 8\pi\mu bU \int_0^\infty \frac{F(\alpha)}{\cosh\alpha\pi} \, d\alpha, \tag{4.3}$$

where

$$F(\alpha) = \int_{t_0}^1 K_\alpha(-\tau)K_\alpha^{(2)}(\tau) \, d\tau \Big/ \int_{t_0}^1 K_\alpha(\tau)K^{(2)}(\tau) \, d\tau. \tag{4.4}$$

With $A(\alpha)$ and $B(\alpha)$ defined by (3.25) and (3.26), Eq. (4.2) gives the well known result for the sphere.

The reviewer has kindly called our attention to the fact that tables of the functions $K_\alpha(t)$ are now being compiled (M. I. Zhurina and L. N. Karmazina, *Tablitsky funktsii Lezhandra* $P_{-1/2+i\tau}(x)$.) These tables should facilitate the computation of the drag coefficient P as a function of ξ_0.

Bibliography

1. H. L. Dryden, F. D. Murnaghan and H. Bateman, *Hydrodynamics*, Bull. No. 84 of the National Research Council, 1932. Reprinted by Dover Publications, New York, 1956
2. L. E. Payne and W. H. Pell, *The Stokes flow problem for a class of axially symmetric bodies*, J. Fluid Mechanics (in print)
3. L. M. Milne-Thomson, *Theoretical hydrodynamics*, 2nd ed., Macmillan Co., New York, 1950
4. A. Weinstein, *Generalized axially symmetric potential theory*, Bull. Am. Math. Soc. 59, 20–38 (1953)
5. L. E. Payne, *Representation formulas for solutions of a class of partial differential equations*, Tech. Note BN-122, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, February, 1958; see also J. Math. Phys. **38**, 145–149 (1959)
6. L. E. Payne, *On axially symmetric flow and the method of generalized electrostatics*, Quart. Appl. Math. **10**,197–204 (1952)
7. E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, University Press, Cambridge, 1931
8. C. Neumann, *Uber die Mehler'schen Kugelfunktionen und deren Anwendung auf elektrostatische Probleme*, Math. Ann. **18**, 195–236 (1881)