MOMENTS OF THE GENERALIZED RAYLEIGH DISTRIBUTION*

BY

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I. Introduction. Gaussian processes are of considerable interest in problems involving random noise. Also of interest is the Rayleigh distribution which arises in work on radar, the detection of signals in noise, etc. [1, 2]. The generalized Rayleigh process promises to be of interest in the future especially when signals in noise are thought of to exist in a finite dimensional Hilbert space [3, 4, 5]. The generalized Rayleigh process was defined and some of its properties were investigated by K. S. Miller, *et al.* [6]. The purpose of this paper is to investigate the moments of the generalized Rayleigh distribution.

Let $X_1, X_2, \cdots X_N$ be independent Gaussian random variables with means $\bar{x}_1$, $\bar{x}_2, \cdots \bar{x}_N$ respectively and equal variances of one. A generalized Rayleigh random variable , Y, (also referred to as a non-central chi-square variable) is defined as

$$Y^2 = \sum_{i=1}^{N} X_i^2 \tag{1.1}$$

and the density function of Y, denoted $g(y)$, is given by [Ref. 6, Eq. 1.6]

$$g(y) = \begin{cases} y_0(y/y_0)^{N/2} \exp\,[(y_0^2 + y^2)/2] I_{(N-2)/2}(y_0 y) & \text{for} \quad y > 0 \\ 0 & \text{for} \quad y \leqq 0, \end{cases} \tag{1.2}$$

where

$$y_0^2 = \sum_{i=1}^{N} \bar{x}_i^2 \tag{1.3}$$

and $I_k(x)$ is the modified Bessel function of the first kind. $g(y)$ is called the generalized Rayleigh distribution. In this paper expressions for the moments about zero of $g(y)$ and several interesting properties of these moments will be derived.

It does not complicate the problem to consider non-integer moments. Therefore, the ath moment of $g(y)$ is given by

$$M_a(N, y_0) = \int_{-\infty}^{\infty} y^a g(y)\, dy, \tag{1.4}$$

where "a" is any real number. (However, as will be seen later, the above integral exists only for a $> -N$ hence a can be any real number greater than $-N$.) Whenever only integral moments are considered the subscript n will be used.

The important results are:

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(1) The power series expression for $M_a(N, y_0)$,

$$M_a(N, y_0) = 2^{a/2} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{\Gamma[r + (N + a)/2]}{r!\,\Gamma[r + (N/2)]} (y_0^2/2)^r, \tag{1.5}$$

(2) The "closed form" expression for $M_a(N, y_0)$,

$$M_a(N, y_0) = 2^{a/2} \exp(-y_0^2/2) \frac{\Gamma[(N + a)/2]}{\Gamma(N/2)} M[(N + a)/2, N/2, y_0^2/2], \tag{1.6}$$

where M (with no subscript) is the confluent hypergeometric function (in the notation of Jahnke and Ende, [7, p. 275]).

(3) The asymptotic expressions for $M_a(N, y_0)$,

$$M_a(N, y_0) \sim y_0^a\left[1 + \frac{a(N + a - 2)}{2y_0^2} + \frac{a(a - 2)(N + a - 2)(N + a - 4)}{2!\,(2y_0^2)^2} + \cdots\right] \quad \text{as} \quad y_0 \to \infty, \tag{1.7}$$

and

$$M_a(N, y_0) \sim N^{a/2} \exp(ay_0^2/2N) \quad \text{as} \quad N \to \infty. \tag{1.8}$$

(4) The recursion formulas

$$M_{a+2}(N, y_0) = (N + a)M_a(N, y_0) + y_0^2\left[M_a(N, y_0) + y_0 \frac{dM_a(N, y_0)}{dy_0}\right] \tag{1.9}$$

and

$$M_{a-2}(N, y_0) = \tfrac{1}{2} \exp(-y_0^2/2)(y_0^2/2)^{(2-N-a)/2} \int_0^{y_0^2/2} \exp(x) x^{(N+a-4)/2} M_a[(2x)^{1/2}]\, dx \quad \text{for} \quad a > 2 - N. \tag{1.10}$$

(5) The upper bounds on negative integer moments

$$M_{-n} \leqq |1/y_0|^n \qquad n = 1, 2, \cdots N - 2 \tag{1.11}$$

and

$$M_{-n} \leqq |1/(N - n)|^{n/2} \qquad n = 1, 2, \cdots N. \tag{1.12}$$

II. General expressions for the moments of the generalized Rayleigh distribution. We first obtain a simple expression for the moments as defined by Eq. (1.4). Substituting (1.2) in (1.4) and abbreviating $M_a(N, y_0)$ by M_a we obtain

$$M_a = y_0^{1-(N/2)} \exp(-y_0^2/2) \int_0^{\infty} y^{(2a+N)/2} \exp(-y^2/2) I_{(N-2)/2}(yy_0)\, dy. \tag{2.1}$$

The integral in the above expression diverges if $a \leqq -N$. Therefore, whenever a is used it will denote a real number greater than $-N$ and n will denote an integer greater than $-N$. Substituting in (2.1) the equivalent power series for $I_{(N+2)/2}(yy_0)$ [8, p. 163], we obtain

$$M_a = 2^{1-(N/2)} \exp(-y_0^2/2) \int_0^{\infty} \sum_{r=0}^{\infty} \frac{y^{N+2r+a-1} \exp(-y^2/2)}{r!\,\Gamma[(N + 2r)/2]} (y_0/2)^{2r}\, dy. \tag{2.2}$$

Interchanging summation and integration in the above, we note that the integral is a gamma function, hence Eq. (1.5).

The sum in (1.5) can, recalling that $B\Gamma(B) = \Gamma(B+1)$, be rewritten,

$$\sum_{r=0}^{\infty} \frac{\Gamma[r+(N+a)/2]}{r!\,\Gamma[r+(N/2)]}(y_0^2/2)^r = \frac{\Gamma[(N+a)/2]}{\Gamma(N/2)} \cdot \left\{1 + \sum_{r=1}^{\infty}\frac{1}{r!}\left[\prod_{k=0}^{r-1}(N+a+2k)/(N+2k)\right](y_0^2/2)^r\right\}. \tag{2.3}$$

The term in brackets in (2.3) is the power series for the confluent hypergeometric function, $M[(N+a)/2, N/2, y_0^2/2]$, and will sometimes be abbreviated by M. Therefore, using (2.3) in (1.5) we obtain the closed form expression (1.6).

III. Asymptotic expressions. In this section asymptotic expressions for $M_a(N, y_o)$ as a function of y_0 with a and N fixed and as a function of N with a and y_0 fixed will be derived.

First consider the case when a and N are fixed. $M_a(N, y_0)$ is given by (1.6) in terms of the hypergeometric function and therefore, using the asymptotic expression for M given on p. 275 of [7], we obtain (1.7), the asymptotic expression for $M_a(N, y_0)$ as $y_0 \to \infty$.

To obtain the asymptotic expression for M_a as $N \to \infty$ we use the power series expression for M_a given in (1.5) where the series has been rewritten as shown in (2.3). The product in the right hand side of (2.3) can be written

$$\prod_{k=0}^{r-1}(N+a+2k)/(N+2k) = [1+(a/N)]^r \prod_{k=0}^{r-1}\frac{(N+2k+a)N}{(N+a)(N+2k)} = [1+(a/N)]^r \prod_{k=0}^{r-1}\left[1 - \frac{2ka}{(N+a)(N+2k)}\right]. \tag{3.1}$$

It is clear from (3.1) that this product is asymptotic to $[1+(a/N)]^r$. Hence from (2.3) and (1.5) we obtain

$$M_a(N, y_0) \sim \frac{\Gamma[(N+a)/2]}{\Gamma(N/2)} 2^{a/2} \exp(ay_0^2/2N) \quad \text{as} \quad N \to \infty. \tag{3.2}$$

But

$$\frac{\Gamma[(N+a)/2]}{\Gamma(N/2)} \sim (N/2)^{a/2} \quad \text{as} \quad N \to \infty \tag{3.3}$$

and hence Eq. (1.8).

IV. Recursion formulas. Recursion formulas can be easily derived for moments of order $a+2$ in terms of the moment of order a and its derivative with respect to y_0. Since $M_0(N, y_0) = 1$, moments of even integer order are easy to compute from this formula. A recursion formula for moments of order $a-2$ is also obtained in terms of an integral involving the moment of order a.

As a preliminary step, consider Eq. (1.5) where x has been used in place of $y_0^2/2$. After differentiating with respect to x and rearranging terms we obtain

$$2x\left(M_a + \frac{dM_a}{dx}\right) = 2\cdot 2^{a/2}\exp(-x)\sum_{r=0}^{\infty}\frac{rx^r}{r!}\frac{\Gamma[r+(N+a)/2]}{\Gamma[r+(N/2)]}. \tag{4.1}$$

But

$$M_{a+2} = 2\cdot 2^{a/2}\exp(-x)\sum_{r=0}^{\infty}\frac{x^r}{r!}\frac{[r+(N+a)/2]\Gamma[r+(N+a)/2]}{\Gamma[r+(N/2)]}. \tag{4.2}$$

Therefore

$$M_{a+2} = (N + a)M_a + 2x\left(M_a + \frac{dM_a}{dx}\right). \tag{4.3}$$

If we put (4.3) in terms of y_0 we obtain the recursion formula given in (1.9). In particular since $M_0 = 1$

$$\begin{aligned} M_2 &= N + y_0^2, \\ M_4 &= N(N + 2) + 2y_0^2(N + 2) + y_0^4, \\ &\text{etc.} \end{aligned} \tag{4.4}$$

To obtain the recursion formulas for decreasing moments we multiply $M_a(x)$ by $\exp(x)x^{(N+a-4)/2}$ and integrate term by term from 0 to $y_0^2/2$ resulting in

$$\int_0^{y_0^2/2} \exp(x)x^{(N+a-4)/2}M_a[(2x)^{1/2}]\,dx = (y_0^2/2)^{(N+a-2)/2}2^{a/2} \sum_{r=0}^{\infty} \frac{\Gamma[r - 1 + (N + a)/2]}{r!\,\Gamma[r + (N/2)]}(y_0^2/2)^r. \tag{4.5}$$

But

$$M_{a-2} = 2^{(a-2)/2} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{\Gamma[r - 1 + (N + a)/2]}{r!\,\Gamma[r + (N/2)]}(y_0^2/2)^r \tag{4.6}$$

which together with (4.5) gives Eq. (1.10). It can be shown that the integration in (4.5) exists only if $a > 2 - N$.

When $a = 0$ (1.10) becomes

$$M_{-2}(N, y_0) = \tfrac{1}{2} \exp(-y_0^2/2)(y_0^2/2)^{(2-N)/2} \int_0^{y_0^2/2} \exp(x)x^{(N-4)/2}\,dx \tag{4.7}$$

and for a few particular values of N we have

$$\begin{aligned} M_{-2}(4, y_0) &= 1/y_0^2, \\ M_{-2}(6, y_0) &= [1 - (2/y_0^2)]/y_0^2 \end{aligned}$$

and

$$M_{-2}(8, y_0) = [1 - (4/y_0^2) + (8/y_0^4)]/y_0^2. \tag{4.8}$$

V. Upper bounds on negative integer moments. Two very simple expressions which are upper bounds on negative integer moments can easily be derived. Let n be a negative even integer, say $n = -2m$, $m = 1, 2, \cdots$ then 1.5 can be written

$$M_{-2m} = 2^{-m} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{1}{r!}(y_0^2/2)^r \prod_{k=1}^{m} 2/(N + 2r - 2k). \tag{5.1}$$

Assuming $(N/2) - m > 0$ the right side of (5.1) is not decreased by letting r be zero in each term of the product resulting in

$$M_{-2m} \leqq 1/(N - 2m)^m. \tag{5.2}$$

Alternately if we assume $(N/2) - 1 \geqq m$ we can replace m for $(N/2) - 1$ in the product in Eq. (5.1) and not decrease the right hand side. Therefore we have

$$M_{-2m} \leqq 2^{-m} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{1}{(r+m)!} (y_0^2/2)^r. \tag{5.3}$$

The sum in the right hand side of (5.3) can be written

$$\sum_{r=0}^{\infty} \frac{1}{r!} (y_0^2/2)^r \frac{r!}{(r+m)!} = (2/y_0^2)^m \sum_{r=m}^{\infty} \frac{1}{r!} (y_0^2/2)^r. \tag{5.4}$$

By filling in some positive terms in the sum on the right hand side of (5.4) it becomes an exponential and we obtain

$$M_{-2m} \leqq 1/y_0^{2m}. \tag{5.5}$$

By Schwartz's inequality

$$M_{m+k}^2 \leqq M_{2m} M_{2k}. \tag{5.6}$$

In particular if $m = 0$ and k is a negative integer, i.e., $k = -n$, $n = 1, 2, \cdots$ then

$$M_{-n}^2 \leqq M_{-2n}. \tag{5.7}$$

Combining (5.7) with (5.5) and (5.2) we obtain (1.11) and (1.12) respectively.

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