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THE IMPERFECTLY CONDUCTING COAXIAL LINE*

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Abstract. In order to determine the range of validity of certain elementary concepts in waveguide theory, the propagation of electromagnetic waves along and through a coaxial line with imperfectly conducting walls is studied in some detail for a particular method of driving. In particular, it is found that the usual concept of attenuation is meaningful only for a certain range of distances from the driving point. Beyond this distance, the electromagnetic field in the coaxial line behaves more like a radiation field. The explanation is supported by the behavior of the electromagnetic field in the imperfect outer conductor of the coaxial line. It is also found that the solution in terms of the "mode" concept has a surprisingly limited region of validity. The reflection coefficient, the radiation pattern and the transverse distribution are also determined.

1. Introduction. The propagation of a wave through an imperfectly conducting coaxial line is of interest from the point of view of waveguide theory. In the elementary theory of waveguides, the wall of the waveguide is assumed to be perfectly conducting. Under this assumption, there arises the extremely important, useful, and convenient concept of "mode." With this concept, these waveguide problems become essentially two-dimensional, since for each mode the dependence on z (the direction of the waveguide) is simply exponential. Furthermore, the number of modes is countably infinite. However, the situation is no longer so simple if it is possible for the electromagnetic wave to reach infinity in a direction perpendicular to the z-axis, as it can when the waveguide is open or when the waveguide has an imperfectly conducting wall. In the latter case, it is sometimes possible to use the idea of an impedance wall to save the situation. Otherwise, the modes lose at least some of their desirable properties, the excellent work of Marcuvitz and coworkers [1] in this connection notwithstanding. As an example of the open waveguide, consider the microstrip. Here the lowest mode, although exponentially decreasing in the direction of the z-axis, is unfortunately exponentially increasing at infinity in any plane z = constant. This is readily understandable and is intimately connected with the infinite length of the line assumed in the theoretical treatment [2]. On the other hand, for any physical waveguide or transmission line, the total length must be finite and consequently any meaningful solution of the physical problem must satisfy the usual radiation condition of Sommerfeld. For

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this reason, the connection between the theoretical and the experimental aspects of the microstrip problem is by no means evident. Furthermore, it is apparently desirable to treat a problem of this type where the generator is not at infinity. No really simple problem of this type seems to have been treated in electromagnetic theory, and the only reasonable candidates seem to be the circular waveguide and the coaxial line. The advantage of the first is that it has one less dimension (the size of the inner conductor), while loosely the advantage of the second is that there is no mixing of the *TE* and the *TM* modes. Mathematically, the second problem is the simpler one of the two, and hence the imperfectly conducting coaxial line is treated in this paper.

The precise geometry of the problem is shown in Fig. 1, where the imperfectly conducting outer conductor of the coaxial line is of finite thickness and runs from $-\infty$ to $+\infty$ along the z-axis while immediately inside a perfect conductor of zero thickness is fitted from $-\infty$ to 0. To avoid meaningless complications, the inner conductor is assumed to be perfectly conducting from $-\infty$ to $+\infty$. It is assumed that a *TEM* wave propagates from left to right; it is incident on the junction at $x = 0$. Thus there is effectively a generator at $z = 0$. This geometry is chosen because, in principle, at least, it possesses a closed solution in terms of quadratures.

Since this problem is treated here only as an example of waveguide theory, there is no necessity of dealing with it in its full generality. Therefore, in the approximate theory of Secs. 5-10, it will be assumed that the free space wavelength is much larger than the transverse dimensions of the coaxial line in addition to the more specialized assumptions (5.1) and (5.2).

This problem is also of considerable interest in connection with the so-called electromagnetic shielding problem. However, in this connection, the assumptions (5.1) and (5.2) are too restrictive, and consequently, a more elaborate approximate theory has to be developed after a thorough understanding of the features of the more restrictive problem treated here has been acquired.

Before studying the problem of the imperfectly conducting coaxial line, it is necessary to know some of the properties of the Green's function to be used in the integral-equation formulation. This Green's function is studied in the next section.

2. Green's function. The interest here is in the implications associated with the finite conductivity of the outer conductor. Since these implications do not depend critically on the choice of the dielectric constant of this imperfect conductor, this dielectric constant is to be identified with that of vacuum in the following consideration. If the conductor is also non-magnetic, then both the magnetic permeability and the dielectric

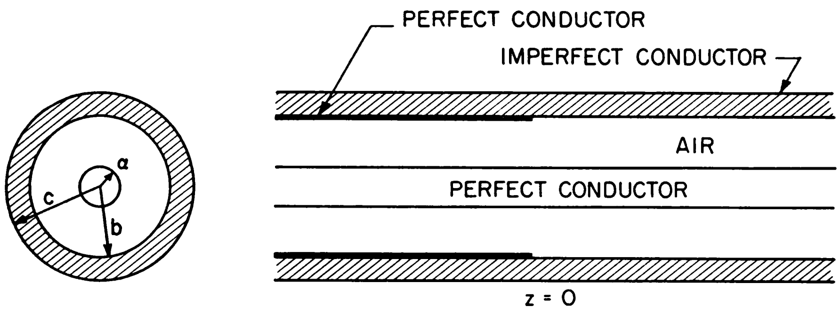

Fig. 1. Geometry of the problem.

constant take those values appropriate for vacuum, say μ and ϵ. With the time dependence exp $(-\, i\omega t)$, Maxwell's equations are

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu\mathbf{H}, \\ \nabla \times \mathbf{H} &= (\sigma - i\omega\epsilon)\mathbf{E} + \mathbf{J}, \\ \nabla \cdot \mathbf{E} &= \nabla\cdot\mathbf{H} = 0, \end{aligned} \tag{2.1}$$

where σ is the conductivity of the imperfect outer conductor of the coaxial line and is replaced by 0 for air; the other symbols all have their usual meanings. If conventional cylindrical coordinates are used, all field quantities are independent of θ in this axially symmetric problem. Therefore

$$E_\theta = H_z = H_r = 0. \tag{2.2}$$

With the continuity of E_z and H_θ as boundary conditions and the radiation condition at infinity, the electromagnetic field is completely determined by (2.1) provided that $\mathbf{J}$ is given. Let $G_0(r; \zeta)$ exp $(i\zeta z)$ be the z-component of the electric field produced by the current source $\mathbf{J} = \mathbf{z}\ \delta(r - b)$ exp $(i\zeta z)$, where $\mathbf{z}$ is the unit vector in the z-direction; it may then be verified that

$$G_0(b, \zeta) = i\omega\mu[k^2\xi^{-1}Q_1 - (k^2 + i\kappa^2)\eta^{-1}Q_2]^{-1}, \tag{2.3}$$

where

$$Q_1 = \frac{H_0^{(2)}(\xi a)H_0^{(1)\prime}(\xi b) - H_0^{(1)}(\xi a)H_0^{(2)\prime}(\xi b)}{H_0^{(2)}(\xi a)H_0^{(1)}(\xi b) - H_0^{(1)}(\xi a)H_0^{(2)}(\xi b)}, \tag{2.4}$$

$$\begin{aligned} Q_2 = &\begin{vmatrix} H_c^{(1)}(\xi c) & H_0^{(1)}(\eta b)H_0^{(2)}(\eta c) - H_0^{(2)}(\eta b)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)}(\eta b)H_0^{(2)\prime}(\eta c) - H_0^{(2)}(\eta b)H_0^{(1)\prime}(\eta c) \end{vmatrix}^{-1} \\ \times &\begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(1)\prime}(\eta b)H_0^{(2)}(\eta c) - H_0^{(2)\prime}(\eta b)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)\prime}(\eta b)H_0^{(2)\prime}(\eta c) - H_0^{(2)\prime}(\eta b)H_0^{(1)\prime}(\eta c) \end{vmatrix}, \end{aligned} \tag{2.5}$$

$$\xi = (k^2 - \zeta^2)^{1/2}, \qquad \eta = (k^2 + i\kappa^2 - \zeta^2)^{1/2}, \tag{2.6}$$

and

$$k^2 = \omega^2\mu\epsilon, \qquad \kappa^2 = \omega\mu\sigma. \tag{2.7}$$

Note that

$$G_0(b, \zeta) = G_0(b, -\zeta), \tag{2.8}$$

and that

$$G_0(b, \zeta) \sim -i(2k^2 + i\kappa^2)^{-1}\omega\mu\zeta \tag{2.9}$$

as $\zeta \to \infty$. Because of (2.9), it is not meaningful to take the Fourier transform of $G_0(b, \zeta)$ with respect to ζ. Since $G_0(b, \zeta) = O(k^2 - \zeta^2)$ as $\zeta \to \pm k$, it is possible to introduce an analog of the ordinary vector potential by defining

$$g_1(b, z) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\zeta G_1(b, \zeta) \exp(i\zeta z), \tag{2.10}$$

where

$$G_1(b, \zeta) = (k^2 - \zeta^2)^{-1} G_0(b, \zeta). \tag{2.11}$$

3. Formulation of the problem. The problem of the coaxial line may now be formulated in terms of a Wiener-Hopf integral equation. Let $i(z)$ be the z-component of the total surface current density on the infinitely thin perfectly conducting sleeve; the incident current density is $\exp(ikz)$. Thus as $z \to -\infty$

$$i(z) \sim \exp(ikz) + \Gamma \exp(-ikz), \tag{3.1}$$

where Γ is the coefficient of reflection with reference to the junction $z = 0$. With the g_1 of (2.10), $i(z)$ satisfies the integral equation

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{0} dz' i(z') g_1(b, z - z') = 0 \tag{3.2}$$

for $z < 0$. If k is assumed to have a small positive imaginary part which eventually approaches zero, then the Fourier transform of (3.2) in the form

$$\int_{-\infty}^{0} dz' \left[\left(\frac{\partial}{\partial z'} - ik\right) i(z')\right]\left[\left(\frac{\partial}{\partial z} + ik\right) g_1(b, z - z')\right] = E_z(b, z) \tag{3.3}$$

is

$$-[(\zeta - k) I(\zeta)][(\zeta + k) G_1(b, \zeta)] = \mathcal{E}_z(b, \zeta). \tag{3.4}$$

This equation is valid in a small strip $|\operatorname{Im} \zeta| < \epsilon$, where $G_1(b, \zeta)$ has no zero. The standard Wiener-Hopf procedure calls for a factorization of G_1 in the form

$$G_1(b, \zeta) = L_+(\zeta)/L_-(\zeta), \tag{3.5}$$

where

$$L_\pm(\zeta) = \exp\left\{(2\pi i)^{-1} \int_{-\infty \mp i\epsilon/2}^{\infty \mp i\epsilon/2} dt (t - \zeta)^{-1} \ln G_1(b, t)\right\}. \tag{3.6}$$

Note that

$$L_-(\zeta) = [L_+(-\zeta)]^{-1}. \tag{3.7}$$

It follows from (3.4) and (3.5) that

$$(k^2 - \zeta^2) I(\zeta) L_+(\zeta) = \mathcal{E}_z(b, \zeta) L_-(\zeta). \tag{3.8}$$

This defines an integral function which is found to be a constant, say C. As a consequence of (3.1), the special case $\zeta = k$ gives

$$C = 2ikL_+(k). \tag{3.9}$$

From (3.8) the electric field is given by

$$E_z(b, z) = (2\pi)^{-1} C \int_{-\infty}^{\infty} d\zeta [L_-(\zeta)]^{-1} \exp(i\zeta z). \tag{3.10}$$

In principle, the problem has been solved in closed form.

4. The radiation pattern. The field far away from the origin is considered here. Since as $\xi \to 0$

$$G_0(b, \zeta) = -(i\omega\epsilon)^{-1}[b \ln(b/2)]\xi^2[1 + O(\xi^4 \ln \xi)], \tag{4.1}$$

$H_\theta(b, z)$ must be proportional to z^{-2} as $z \to \infty$. This fact is of interest because it shows immediately that, as $z \to \infty$, an infinite number of "modes" must contribute significantly. The reason is that the combination of a finite number of modes must yield an exponential decay in the z-direction. This z^{-2} decay may be understood as follows. Because the outer conductor of the coaxial line is not perfectly conducting, there is a coupling of the field inside the coaxial line with the radiation field outside of the coaxial line. Since the outside field decays as z^{-2} because of cylindrical symmetry, the field inside the coaxial line must also decay as z^{-2}.

Thus the term "radiation pattern" is meaningful for the present problem. It follows from (3.8) and (3.9) that, as $r, z \to \infty$ with fixed $\phi = \tan^{-1}(r/z)$,

$$H_\theta(r, z) \sim A(\phi)(r^2 + z^2)^{-1/2} \exp [ik(r^2 + z^2)^{1/2}], \tag{4.2}$$

where

$$\begin{aligned} A(\phi) = \frac{8i\omega\epsilon}{\pi^2 c\Phi} \frac{L_+(k)}{L_-(k \cos \phi)} \{H_0^{(1)}(kc \sin \phi) \sin \phi[H_0^{(1)}(b\Phi)H_0^{(2)\prime}(c\Phi) - H_0^{(2)}(b\Phi)H_0^{(1)\prime}(c\Phi)] \\ - k(k^2 + i\kappa^2)^{-1}\Phi H_0^{(1)\prime}(kc \sin \phi)[H_0^{(1)}(b\Phi)H_0^{(2)}(c\Phi) - H_0^{(2)}(b\Phi)H_0^{(1)}(c\Phi)]\}^{-1}, \end{aligned} \tag{4.3}$$

with

$$\Phi = (k^2 \sin^2 \phi + i\kappa^2)^{1/2}. \tag{4.4}$$

5. Approximations for small skin depth. Although the exact solution has been obtained in principle, it is not very useful because of its complexity. The situation is greatly simplified if the coaxial line is only slightly leaky as when $c - b$ is much larger than skin depth $\delta = \kappa^{-1}\sqrt{2}$, or more precisely, if

$$b \gg \delta \tag{5.1}$$

and

$$\exp [\kappa(c - b) \sqrt{2}] \gg 1. \tag{5.2}$$

Under these assumptions, Q_2 of (2.5) is approximately

$$Q_2^{(1)} = i. \tag{5.3}$$

Furthermore when $kc \ll 1$, (5.3) leads to the following approximation of $G_0(b, \zeta)$

$$G_0^{(1)}(b, \zeta) = i\eta\sigma^{-1}[1 + C_1(k/\xi)^2]^{-1}, \tag{5.4}$$

where

$$C_1 = e^{i\pi/4}[kb \ln (b/a)]^{-1}. \tag{5.5}$$

The physical meaning of (5.2) is as follows. When the coaxial line is lossless, the characteristic scale of length for the variation of the field vectors in the z-direction is k^{-1}, which is very large. When the coaxial line is slightly leaky, the field inside the line tends to decrease when z increases, and this characteristic scale of length is the smaller one of k^{-1} and $\kappa b k^{-1} \exp [\kappa(c - b) \sqrt{2}]$. If this characteristic length is much larger than the outer radius c, then the transverse problem is essentially separated from the longitudinal problem, except possibly near $z = 0$. Once the longitudinal and the transverse problems are separated, the Green's function G_0 must admit simplification, and basically only the longitudinal problem need be considered. The condition (5.2) implies that the frequency under consideration should not be too low in order that the thickness of the imperfect outer conductor be several times the skin depth.

When k^2 is neglected compared with κ^2, (5.4) and (2.11) lead to

$$G_1^{(1)}(b, \zeta) = i\sigma^{-1}(i\kappa^2 - \zeta^2)^{1/2}[k^2(1 + C_1) - \zeta^2]^{-1}. \tag{5.6}$$

This may be factored by inspection to give

$$L_+^{(1)}(\zeta) = (i/\sigma)^{1/2}(e^{i\pi/4}\kappa + \zeta)^{1/2}[k(1 + C_1)^{1/2} + \zeta]^{-1} \tag{5.7a}$$

and

$$L_-^{(1)}(\zeta) = (i/\sigma)^{-1/2}(e^{i\pi/4}\kappa - \zeta)^{-1/2}[k(1 + C_1)^{1/2} - \zeta]. \tag{5.7b}$$

In particular, it follows from (3.9) and (3.8) respectively that

$$C^{(1)} = -\kappa^{1/2}\sigma^{-1/2}e^{-i\pi/8}[1 - C_1/4] \tag{5.8}$$

and

$$\mathcal{E}_z^{(1)}(b, \zeta) = -e^{i\pi/8}\kappa^{1/2}\sigma^{-1}[1 - C_1/4][e^{i\pi/4}\kappa - \zeta]^{1/2}[k(1 + C_1)^{1/2} - \zeta]^{-1}. \tag{5.9}$$

Note that $\mathcal{E}_z$ is proportional to σ^{-1} as expected.

The advantage of this approximation is that the factorization can be carried out explicitly. However, the exact Green's function $G_0(b, \zeta)$ as given by (2.3) has branch points at $\zeta = \pm k$, while the function $G_0^{(1)}(b, \zeta)$ as given by (5.4) has no branch points there. Since it is precisely these branch points that determine the behavior of the field vectors for large z, this first approximation is not good enough. Since, as $\xi \to 0$, the series expansion of the Hankel function yields that $| H_0^{(1)}(\zeta c)/H_0^{(1)\prime}(\xi c) | = O(\xi \ln \xi)$, a better approximation of Q_2 in (2.5) is

$$Q_2^{(2)} = i(1 + P), \tag{5.10}$$

where

$$P = 4H_0^{(1)}(\xi c)e^{2i\eta(c-b)}[\eta\xi^{-1}(k/\kappa)^2 H_0^{(1)\prime}(\xi c) - H_0^{(1)}(\xi c)]. \tag{5.11}$$

The term P is important only when ξ is small. Therefore it is sufficient to use

$$L_+^{(2)}(k) = L_+^{(1)}(k), \quad \text{or} \quad C^{(2)} = C^{(1)}. \tag{5.12}$$

Let

$$M(\zeta) = P[1 + C_1(k/\xi)^2]^{-1}, \tag{5.13}$$

then (5.10) leads to

$$G_1^{(2)}(b, \zeta) = G_1^{(1)}(b, \zeta)[1 + M(\zeta)]^{-1}. \tag{5.14}$$

Let the quantity in the bracket of (5.14) be factored in the form

$$[1 + M(\zeta)]^{-1} = [1 + N_+(\zeta)]/[1 + N_-(\zeta)], \tag{5.15}$$

where

$$\begin{aligned} N_\pm(\zeta) &= -1 + \exp\left\{-(2\pi i)^{-1}\int_{-\infty\mp i\epsilon/2}^{\infty\mp i\epsilon/2} dt(t - \zeta)^{-1}\ln[1 + M(t)]\right\} \\ &= -(2\pi i)^{-1}\int_{-\infty\mp i\epsilon/2}^{\infty\mp i\epsilon/2} dt(t - \zeta)^{-1}M(t). \end{aligned} \tag{5.16}$$

It follows from (5.15) that

$$L_{\pm}^{(2)}(\zeta) = L_{\pm}^{(1)}(\zeta)[1 + N_{\pm}(\zeta)]. \tag{5.17}$$

In the next few sections, the electromagnetic field at various points of the space will be studied. For simplicity all superscripts (1) and (2) will be omitted and only leading terms retained since, except at great distances from the junction at $z = 0$, the first approximation and the second approximation differ negligibly.

6. Approximate field inside the coaxial line. In this section, the behavior of the field for $a \leq r \leq b$ is considered. For $a \leq r \leq b$, let

$$S_1(r, \zeta) = \frac{H_0^{(1)}(\xi a)H_0^{(2)}(\xi r) - H_0^{(2)}(\xi a)H_0^{(1)}(\xi r)}{H_0^{(1)}(\xi a)H_0^{(2)}(\xi b) - H_0^{(2)}(\xi a)H_0^{(1)}(\xi b)}, \tag{6.1}$$

so that

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta)S_1(r, \zeta). \tag{6.2}$$

Note that as $\xi \to 0$,

$$S_1(r, \zeta) \sim \ln (r/a)/\ln (b/a). \tag{6.3}$$

Equation (6.3) implies that, when $z \gg b$, the transverse distribution of E_z is essentially $\ln(r/a)$.

The region $a \leq r \leq b$ may be divided into various subregions I $-$ V as follows

Region I: $-z \gg b$;

Region III: $z \gg b$ but $kz \mid C_1 \mid \ll 1$, and

Region V: $k \mid C_1 \mid z \gg 1$.

Region II is the region between regions III and I; and

Region IV is the region between regions III and V.

The region II is the junction region and will not be studied further.

Let region I be considered first. In this region, E_z is obviously very small. Thus, it is necessary to consider $H_\theta(r, z)$. It follows from (6.2) that the Fourier transform of $H_\theta(r, z)$ is

$$\mathcal{H}_\theta(r, \zeta) = i\omega\epsilon\xi^{-2}\mathcal{E}_z(b, \zeta)(\partial/\partial r)S_1(r, \zeta). \tag{6.4}$$

If Res denotes "residue of", then in this region (3.1) holds with

$$\Gamma = -[\operatorname*{Res}_{\zeta=-k} \xi^{-2}\mathcal{E}_z(b, \zeta)]/[\operatorname*{Res}_{\zeta=k} \xi^{-2}\mathcal{E}_z(b, \zeta)] = C_1/4. \tag{6.5}$$

Thus

$$H_\theta(r, z) = \frac{r}{b}[e^{ikz} + \tfrac{1}{4}C_1e^{-ikz}]. \tag{6.6}$$

For region III, it follows from (5.9), (6.2) and (6.3) that

$$E_z(r, z) = e^{-i\pi/4}\kappa\sigma^{-1}\frac{\ln (r/a)}{\ln (b/a)}e^{ikz}\exp\left[e^{i3\pi/4}\frac{kz}{\kappa b \ln (b/a)}\right], \tag{6.7}$$

and

$$H_\theta(r, z) = \frac{b}{r}e^{ikz}\exp\left[e^{i3\pi/4}\frac{kz}{\kappa b \ln (b/a)}\right]. \tag{6.8}$$

Therefore, the conventional theory of waveguides with a slightly lossy wall is valid provided that $kz \mid C_1 \mid \ll 1$ where C_1 is defined in (5.5).

For the region V, the second approximation has to be used. The substitution of (5.17) into (3.8) gives

$$F^{(2)}(\zeta) = F^{(1)}(\zeta)[1 - N_-(\zeta)]. \tag{6.9}$$

When z is very large, only a small region in the neighborhood of $\zeta = k$ can contribute to the inverse Fourier transform. Therefore, it is only necessary to consider the second term $-F^{(1)}(\zeta)N_-(\zeta)$, since $F^{(1)}(\zeta)$ is analytic at $\zeta = k$. With this notion, $N_-(\zeta)$ may be found explicitly near $\zeta = k$ as follows:

$$N_-(\zeta) = M(\zeta), \tag{6.10}$$

since $N_+(\zeta)$ is analytic at $\zeta = k$. On the other hand, it follows from (5.13) that, near $\zeta = k$,

$$M(\zeta) = \xi^4[k^2(1 + C_1) - \zeta^2)]^{-1}4\kappa ck^{-2}e^{-i\pi/4} \ln (\xi c) \exp [2i\eta_0(c - b)], \tag{6.11}$$

where $\eta_0 = \kappa \exp (i\pi/4)$. Consequently, for this region

$$\begin{aligned} F^{(2)}(\zeta) &= -F^{(1)}(\zeta)M(\zeta) \\ &= 2\kappa^2k^{-3}\sigma^{-1}[k(1 + C_1/2) - \zeta]^{-2}\, \xi^4c \ln (\xi c) \exp [2i\eta_0(c - b)]. \end{aligned} \tag{6.12}$$

The substitution of (6.3) and (6.12) into (6.2) now yields

$$E_z(r, z) = \frac{c}{\pi}\kappa^2k^{-3}\sigma^{-1} \exp [2i\eta_0(c - b)] \frac{\ln (r/a)}{\ln (b/a)} F, \tag{6.13}$$

where

$$F = \int_{-\infty}^{\infty} d\zeta e^{i\zeta z}[k(1 + C_1/2) - \zeta]^{-2}(k^2 - \zeta^2)^2 \ln [(k^2 - \zeta^2)^{1/2}c]. \tag{6.14}$$

The integral in (6.14) may be evaluated by closing the contour of integration in the upper half plane. Thus

$$\begin{aligned} F = -4\pi iC_1^2k^3[1 - 2\pi i + 2 \ln (k^2C_1c^2)] - 2\pi zC_1^2k^4[\ln (k^2C_1c^2) - i\pi] \\ + 4\pi ik^2e^{ikz} \int_0^{\infty} d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2}\zeta^2. \end{aligned} \tag{6.15}$$

The first two terms on the right hand of (6.15) are never large enough to make a significant contribution so that they may be omitted. Thus

$$E_z(r, z) = 4ic\kappa^2k^{-1}\sigma^{-1}e^{ikz} \{\exp [2i\eta_0(c - b)]\} \frac{\ln (r/a)}{\ln (b/a)} \int_0^{\infty} d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2}\zeta^2. \tag{6.16}$$

As $z \to \infty$ a stationary phase integration yields

$$E_z(r, z) \sim -32i\kappa^4cb^2k^{-3}\sigma^{-1}e^{ikz}\{\exp [2i\eta_0(c - b)]\} \ln (b/a) \ln (r/a)z^{-3} \tag{6.17}$$

and

$$H_\theta(r, z) = 8i\kappa^{-2}cb^2k^{-2}e^{ikz}\{\exp [2i\eta_0(c - b)]\} \ln (b/a) \frac{1}{r}z^{-2}. \tag{6.18}$$

Note that E_z behaves like z^{-3}, and hence H_θ and E_r behave like z^{-2}. It is interesting to note that the asymptotic phase velocity is that of vacuum.

Finally, there is still region IV to be studied. Because of the form of Eq. (6.9), it is only necessary to add (6.7) and (6.16). Thus

$$E_z(r, z) = \Big\{e^{-i\pi/4} \exp\left(\frac{e^{i3\pi/4}kz}{\kappa b \ln(b/a)}\right) + 4ic\kappa k^{-1}\{\exp[2i\eta_0(c-b)]\} \int_0^\infty d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2} \zeta^2\Big\} \kappa\sigma^{-1} e^{ikz} \frac{\ln(r/a)}{\ln(b/a)}. \tag{6.19}$$

This is an approximate formula valid for all z so that $z \gg b$. It gives explicitly the correction to the simple theory (given by the first term) for waveguides with leaky walls.

These explicit results varify in detail the discussion at the beginning of Sec. 4.

7. Approximate field in the imperfect conductor. The region occupied by the imperfect conductor may also be divided into five subregions, I′ − V′, as given in the last section except that here $b \le r \le c$. In region III′, the conventional waveguide theory with exponential dependence of z must hold again. Thus attention is concentrated on the region V′.

The function that corresponds to the S_1 of (6.1) is now

$$S_2(r, \zeta) = D(r)/D(b), \tag{7.1}$$

where

$$D(r) = \begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(1)}(\eta r)H_0^{(2)}(\eta c) - H_0^{(2)}(\eta r)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)}(\eta r)H_0^{(2)\prime}(\eta c) - H_0^{(2)}(\eta r)H_0^{(1)\prime}(\eta c) \end{vmatrix}. \tag{7.2}$$

Analogous to (6.2), it follows that, for $b \le r \le c$,

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta)S_2(r, \zeta). \tag{7.3}$$

For the present purpose, the appropriate approximate formula for S_2 is

$$S_2(r, \zeta) = (b/r)^{1/2}\{\exp[i\eta_0(r-b)]\}(1 - \exp[2i\eta_0(c-r)] - 2e^{-i\pi/4}\{\exp[2i\eta_0(c-r)] - \exp[2i\eta_0(c-b)]\}\kappa c(\xi/k)^2 \ln(\xi c)) \tag{7.4}$$

when ξ/k is small. With (7.3) written in the form

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z^{(1)}(b, \zeta)[1 - M(\zeta)]S_2(r, \zeta), \tag{7.5}$$

the electromagnetic field in region V′ is found to be

$$E_z(r, z) = 4\kappa^{-3}cb\sigma^{-1}k^{-3}(b/r)^{1/2} \ln(b/a)e^{ikz} \exp[i\eta_0(r-b)](e^{i\pi/4}k\{\exp[2i\eta_0(c-r)] - \exp[2i\eta_0(c-b)]\}z^{-2} - 8i\kappa b \ln(b/a) \exp[2i\eta_0(c-b)]z^{-3}), \tag{7.6}$$

and, with the second term in the parentheses neglected

$$H_z(r, z) = 4i\kappa^2cbk^{-2}(b/r)^{1/2} \ln(b/a)e^{ikz}\{\exp[i\eta_0(r-b)]\}z^{-2} \cdot\{\exp[i\eta_0(c-r)] + \exp[i\eta_0(c+r-2b)]\}. \tag{7.7}$$

Here the second term in the brackets evidently represents a reflection at $r = b$. It is noted that (7.6) and (6.17) give the same value for E_z at $r = b$, while (7.7) and (6.18) give the same value for H_θ at $r = b$. This should be the case for a self-consistent scheme of approximation.

A quanity of particular interest is the transverse distribution of the magnetic field in the outer conductor. For region III′, it is

$$| H_\theta(r, z) | \sim r^{-1/2}\left[\exp\frac{-(r-b)}{\delta} + \exp\frac{-(2c-b-r)}{\delta}\right], \tag{7.8}$$

while for region V′, it is

$$| H_\theta(r, z) | \sim r^{-1/2}\left[\exp\frac{-(c-r)}{\delta} + \exp\frac{-(c+r-2b)}{\delta}\right]. \tag{7.9}$$

These distributions indicate clearly that, for regions III and III′, the electromagnetic energy effectively leaks from the coaxial line to the radiation field, while for regions V and V′, the electromagnetic energy effectively leaks from the radiation field into the coaxial line. This lends further support to the qualitative discussion of Sec. 4. For region IV′, the distribution of the magnetic field must be intermediate between that given by (7.8) and that of (7.9). These results are schematically sketched in Fig. 2.

8. Approximate radiation field for large z. It remains to consider the situation outside of the coaxial line where the structure of the field is enormously complicated. In this section, attention is restricted to the extension of regions V and V′, namely the region V″: $kz \mid C_1 \mid \gg 1$ and $r > c$. This part of the calculation is therefore a continuation of that of the last section. In the next section the radiation pattern is to be found approximately, as a continuation of Sec. 4. In Sec. 10, attention is turned to the extension III″ of regions III and III′. There the relation with the lowest "mode" of the coaxial line is evident, and the present theory puts a limit on the range of validity of this "mode" solution.

For $r > c$ and according to (2.3), the function that corresponds to S_1 and S_2 is given by

$$S_3(r, \zeta) = \frac{4H_0^{(1)}(\xi r)}{\pi i \eta c\, D(b)}, \tag{8.1}$$

and E_z is determined by

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta) S_3(r, \zeta). \tag{8.2}$$

The situation here differs from the previous ones in that $H_0^{(1)}(\xi r)$, which is a factor in

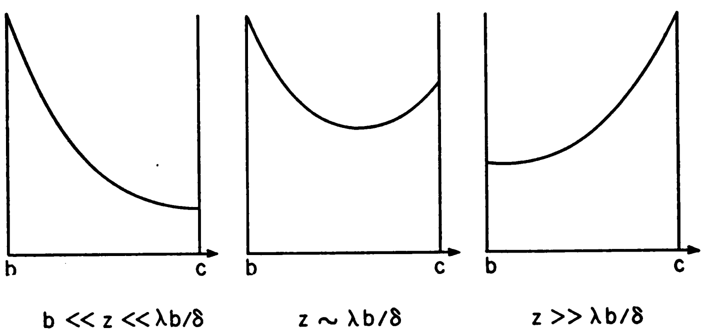

FIG. 2. Sketches of magnetic field distribution in the outer conductor.

(8.1), has branch points at $\zeta = \pm k$. Furthermore, other contributions with branch points, from the determinant and from $F(\zeta)$, have at least a factor ξ^2. For region V″, these other contributions may be neglected. Therefore, (5.9) is adequate for use in (8.2) with the result

$$\mathcal{E}_z(r, \zeta) = -2\pi e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-3} b^{3/2} c^{1/2} \xi^2 \ln (b/a)\{\exp [i\eta_0(c - b)]\} H_0^{(1)}(\xi r). \qquad (8.3)$$

From the discontinuity of the Hankel function across the branch cut, the inverse Fourier transform gives

$$E_z(r, z) = -2e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-3} b^{3/2} c^{1/2} \ln (b/a)\{\exp [i\eta_0(c - b)]\} e^{ikz} r^{-4} B\left(\frac{z}{2kr^2}\right), \qquad (8.4)$$

where

$$B(x) = \int_0^\infty s^3 \, ds J_0(s) \exp (-is^2 x). \qquad (8.5)$$

It does not seem possible to evaluate $B(x)$ explicitly in terms of known functions. However, it is easy to find that, as $x \to \infty$

$$B(x) \sim -\tfrac{1}{2} x^{-2}. \qquad (8.6)$$

The substitution of (8.6) into (8.4) yields

$$E_z(r, z) = 4e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-2} b^{3/2} c^{1/2} \ln (b/a)\{\exp [i\eta_0(c - b)]\} e^{ikz} z^{-2}. \qquad (8.7)$$

This result is valid for $z \gg b/k\delta$ and $c \le r \ll (z/k)^{1/2}$. Note that this field is independent of r, as must be the case for a far-sone radiation field. Also note that (8.7) is consistent with (7.6).

It is not possible to find $H_\theta(r, z)$ directly from (8.7). To find H_θ, it is necessary to go back to (8.1), the reason being that $\partial/\partial r$ nullifies the leading term in the present calculation. The determination of the magnetic field is not given here since it does not lead to anything interesting.

9. The approximate radiation pattern. In this section, (4.3) is to be simplified under the assumptions (5.1) and (5.2). First, the functions $L_+(k)$ and $L_-(k \cos \phi)$ may be eliminated by (3.8), (3.9) and (5.9). The rest of the calculation is straightforward with the result

$$A(\phi) = k^2\kappa^{-2} b^{1/2} c^{1/2}\{\exp [i\eta_0(c - b)]\}[(1 + C_1)^{1/2} - \cos \phi]^{-1} \sin \phi \cdot\left[1 - e^{-i\pi/4}\kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2} + \frac{\pi}{2} e^{i\pi/4} \kappa c \sin^2 \phi\right]^{-1}, \qquad (9.1)$$

where $\gamma = 1.78107$. Note that $A(0) = 0$, as it should.

It is interesting to know the direction ϕ_0 of the major lobe for this radiation pattern. This may be found as follows. From the last factor in (9.1), it is seen that

$$\kappa c \sin^2 \phi_0 \ln \frac{\gamma k c \sin \phi_0}{2} = O(1). \qquad (9.2)$$

Therefore, in the vicinity of ϕ_0, $A(\phi)$ is approximately proportional to

$$A(\phi) \sim \text{const} \sin \phi \left[1 - e^{-i\pi/4}\kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2}\right]^{-1}. \qquad (9.3)$$

The absolute value of this is

$$| A(\phi) |^2 \sim \text{const} \sin^2 \phi \left[1 - \sqrt{2}\, \kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2} + \kappa^2 c^2 \sin^4 \phi \left(\ln \frac{\gamma k c \sin \phi}{2} \right)^2 \right]^{-1}. \tag{9.4}$$

This quantity on the right has a maximum at approximately

$$\phi_0 = \left(\frac{1}{2} \kappa c \ln \frac{\kappa}{k^2 c} \right)^{-1/2}, \tag{9.5}$$

whence

$$| A(\phi_0) | = \kappa^{-1} k^2 b^{3/2} c^{1/2} \left[\exp \frac{-(c - b)}{\delta} \right] \ln (b/a)[1 + 2^{-1/2}]^{1/2}. \tag{9.6}$$

10. The "mode." Finally, consider the region III″ defined by $z \gg b$, $kz \mid C_1 \mid \ll 1$ and $r > c$. For this region (5.9) may be used for $F(\zeta)$. Therefore, from (8.1) and (8.2) it follows that

$$\mathcal{E}_z(r, \zeta) = -e^{i\pi/4} \kappa \sigma^{-1} [k(1 + C_1)^{1/2} - \zeta]^{-1} \frac{4 H_0^{(1)}(\xi r)}{\pi i \eta c} \left[H_0^{(1)}(\eta b) \begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(2)}(\eta c) \\ \frac{\eta}{\xi} \frac{k^2}{k^2 + i\kappa^2} H_0^{(1)\prime}(\xi c) & H_0^{(2)\prime}(\eta c) \end{vmatrix} \right]^{-1}, \tag{10.1}$$

and hence

$$\mathcal{H}_z(r, \zeta) = 2 e^{i\pi/4} k^2 \kappa^{-1} \xi^{-1} [k(1 + C_1)^{1/2} - \zeta]^{-1} (b/c)^{1/2} H_1^{(1)}(\xi r) \{ \exp [i \eta_0 (c - b)] \} \cdot \left\{ -i + \frac{2}{\pi} \ln \frac{\gamma \xi c}{2} - \frac{2}{\pi} \frac{k^2 \eta_0}{\kappa^2 \xi^2 c} \right\}^{-1}. \tag{10.2}$$

In particular, when $kr \ll 1$, $\mathcal{H}_z(r, \zeta)$ and hence $H_z(r, \zeta)$ are proportional to $1/r$. This fact agrees with the result given for the lowest "mode." Secondly, when r increases, the phase of $H_1^{(1)}(\xi r)$ cannot decrease near $\zeta = k$, and hence the phase of H_z increases. This again agrees with the result for the lowest "mode" [2]. However, beyond that, the result for the lowest "mode" does not agree quantitatively with (10.2). In other words, for the present method of driving, the lowest "mode" does not dominate anywhere outside of the coaxial line. The "mode" picture therefore has very limited scope of application indeed.

11. Summary and conclusions. The problem of the imperfectly conducting coaxial line driven as shown in Fig. 1 has been studied in some detail. The results for small skin depth may be summarized qualitatively as follows. The place where more detail may be found is written in square brackets.

A. The reflection coefficient in the line at $z = 0$ is of the order $\delta / b \ln(b/a)$ [(6.5)].

B. For $r < c$ and $b \ll z \ll \lambda b/\delta$ the electromagnetic field decays exponentially [(6.7) and (6.8)].

C. In the dielectric inside the coaxial line, i.e., $a \le r \le b$, and for $z \gg \lambda b/\delta$, the longitudinal field decays as z^{-3} but the transverse fields decay as z^{-2} [(6.17) and (6.18)].

D. In the outer conductor of the coaxial line and for $z \gg \lambda b/\delta$, all field components decay as z^{-2} [(7.6) and (7.7)].

E. In the dielectric inside the coaxial line, the transverse distribution of each field component is essentially independent of z for all $z \gg b$[(6.19)].

F. In the outer conductor the transverse distributions are not even approximately independent of z [(7.8), (7.9) and Fig. 2].

G. Outside the coaxial line for $z \gg \lambda b/\delta$ and $c \leq r \ll (z/k)^{1/2}$, the field component E_z is approximately independent of r[(8.7)].

H. The radiation pattern has a maximum in a direction slightly inclined from the direction of the coaxial line [(9.5) and (9.6)].

I. The "mode" picture has only qualitative meaning outside of the coaxial line [Sec. 10].

It is to be expected that many of these statements are true for any waveguide with a leaky wall and driven in a similar fashion. Let D_1 be the region enclosed by the waveguide wall, D_2 be the region occupied by the waveguide wall, and $D_3 = D_1 + D_2$. Also let L be a typical transverse dimension of the waveguide, then the following generalizations may be conjectured.

B'. The electromagnetic field decays exponentially for $L \ll z \ll \lambda L/\delta$ in the region D_3.

C'. Assume $z \gg \lambda L/\delta$. Then for a *TEM* incident wave, the longitudinal field decays as z^{-3} but the transverse fields decay as z^{-2} in the region D_1. For a non-*TEM* incident wave, all components decay as z^{-2}.

D'. For $z \gg \lambda L/\delta$ and in the region D_2, all field components decay as z^{-2}.

E' In the region D_1, the transverse distribution of the field components are essentially the same for all $z \gg L$.

F'. The statement in *E'* is not true for D_2.

H'. The radiation pattern has a maximum in a direction slightly inclined from the direction of the waveguide. (This seems to be true for both fast and slow waves.)

I'. The "mode" picture has very limited validity in general.

The validity of these conjectures remains to be investigated. In particular, their correctness for a non-*TEM* incident wave should not be taken for granted.

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