ON OPTIMUM RECTANGULAR COOLING FINS*

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1. Introduction. In the past, most discussions concerning the cooling fins of a heat exchanger were confined to cases where the heat from the fin surface was eventually dissipated by the surrounding convective fluid. For this case, the governing equation is linear and the solution can be obtained without difficulty [1]. However, as space technology advances, a heat exchanger may have to be designed for an environment where the only heat transfer mechanism is by radiation. Furthermore, in any space vehicle design, the over-all weight of the vehicle is of utmost importance. It is, therefore, desirable to know the fin geometry of least weight. The essential difficulty in dealing with cooling fins when the convective transfer mechanism becomes insignificant arises from the fact that the governing equation is no longer linear. Up to the present, the solutions for this type of problem were obtained mainly by numerical method [2, 3]. It is difficult to optimize the result thus obtained.

The present paper presents a parametric solution for a rectangular cooling fin in terms of known functions, from which the optimum geometry of the fin with least weight is uniquely determined.

2. Statement of the problem. The governing differential equation of the temperature $T(x)$ in a rectangular thin fin as shown in Fig. 1 is

$$\frac{d^2T}{dx^2} - \frac{2s}{kb} T^\alpha = 0, \tag{1}$$

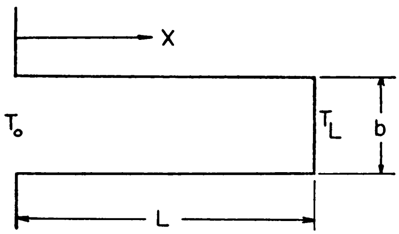

FIG. 1

where

s = heat transfer coefficient,
k = conductivity of the fin material,
b = fin width,
α = a constant, equal to $1 \rightarrow 4$ in actual application and assumed to be greater than unity in the present problem.

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The boundary conditions for a long fin are

$$T = T_0 \quad \text{at} \quad x = 0 \tag{2}$$

$$\frac{dT}{dx} = 0 \quad \text{at} \quad x = L. \tag{3}$$

The cooling rate of the fin in terms of the rate of heat conducted out of the fin base is

$$q = -kb \frac{dT}{dx}\bigg|_{x=0}. \tag{4}$$

The problem may now be stated as: For a fixed fin weight (or area $A = bL$), determine the value of b such that q is a maximum subject to the governing equation (1) and the boundary conditions (2) and (3).

3. The temperature distribution. Though the solution of the system of Eqs. (1), (2) and (3) can not be obtained in closed form, it is possible to derive a parametric relation between T and x as shown below.

Multiplying Eq. (1) by dT/dx and integrating gives

$$\left(\frac{dT}{dx}\right)^2 - \frac{4s}{(\alpha + 1)kb} T^{\alpha+1} + C = 0. \tag{5}$$

The constant C is determined by Eq. (3) and by assuming

$$T = T_L \quad \text{at} \quad x = L. \tag{6}$$

Then,

$$C = \frac{4s}{(\alpha + 1)kb} T_L^{\alpha+1}. \tag{7}$$

Note that the boundary value problem is completely defined by Eqs. (1), (2) and (3). The additional condition, Eq. (6), is used merely as a parameter which shall be uniquely determined. From Eqs. (5) and (7)

$$\frac{dT}{dx} = -\left[\frac{4s}{(\alpha + 1)kb}\right]^{1/2} T^{(\alpha+1)/2}\left[1 - \left(\frac{T_L}{T}\right)^{\alpha+1}\right]^{1/2}. \tag{8}$$

Introducing the new variables

$$t = (T_L/T)^{\alpha+1}, \tag{9}$$

$$t_0 = (T_L/T_0)^{\alpha+1}. \tag{10}$$

Equation (8) becomes

$$t^{\beta-1}(1 - t)^{-1/2} \frac{dt}{dx} = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}, \tag{11}$$

where $\beta = (\alpha - 1)/2(\alpha + 1)$.

Integrating Eq. (11) gives

$$\int_t^1 \eta^{\beta-1}(1 - \eta)^{-1/2}\, d\eta = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2} \int_x^L dx \tag{12}$$

or

$$B(\beta, \tfrac{1}{2}) - B_t(\beta, \tfrac{1}{2}) = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}(L - x), \tag{13}$$

where B is the complete Beta function and B_t, the incomplete Beta function which is defined for $0 \leq t \leq 1$ and $\beta > 0$ [4]. The two conditions for the existence of the Beta function will be examined with regard to Eq. (13):

(i) when $t = 0$, Eq. (13) fails to give a meaningful solution. However, for this case, either $T \to \infty$ which is physically impossible or $T_L = 0$ which will be discussed at the last section as a special case. For the present discussion, we assume $0 < t \leq 1$ in Eq. (13).

(ii) when $\beta > 0$, $\alpha > 1$. If $\alpha = 1$, the problem is for the case with the ordinary convective surface condition, for which the solution is available [1].

Under the restrictions of $0 < t \leq 1$ and $\alpha > 1$, Eq. (13) expresses the functional relation between T and x with T_L as the parameter. Furthermore, the values of T and x have a one-to-one correspondence, since both sides of Eq. (13) are single-valued functions. To determine the parameter T_L, Eq. (2) is substituted into Eq. (13).

$$B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) = \left[\frac{4(\alpha + 1)sL^2}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}. \tag{14}$$

After T_L is calculated from Eq. (14) for a fixed physical condition, the rate of heat transfer, Eq. (4), becomes

$$q = \left[\frac{4skb}{(\alpha + 1)}\right]^{1/2} (T_0^{\alpha+1} - T_L^{\alpha+1})^{1/2}. \tag{15}$$

4. The optimum geometry. Equation (15) gives the rate of heat transfer in terms of the parameter T_L which has to be determined from Eq. (14). For a fixed $A = bL$, T_L will vary with b, so that q is, in general, a function of b and T_L. If Eqs. (14) and (15) are rewritten as

$$q(b, T_L) = \left(\frac{4sk}{\alpha + 1}\right)^{1/2} b^{1/2}(T_0^{\alpha+1} - T_L^{\alpha+1})^{1/2}, \tag{16}$$

$$p(b, T_L) = B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) - \left[\frac{4(\alpha + 1)sA^2}{k}\right]^{1/2} b^{-3/2} T_L^{(\alpha-1)/2} = 0, \tag{17}$$

where q and p are now to be considered as functions of b and T_L, the equivalent problem of optimizing q will be to find the extreme values of the function $q(b, T_L)$ subject to the subsidiary condition $p(b, T_L) = 0$. The stationary point thus found gives the desired optimum values of b and T_L. The solution to this problem can be obtained by means of Lagrange's multiplier m defined by [5]

$$\frac{\partial q}{\partial b} + m \frac{\partial p}{\partial b} = 0, \tag{18}$$

$$\frac{\partial q}{\partial T_L} + m \frac{\partial p}{\partial T_L} = 0. \tag{19}$$

Equations (17), (18) and (19) serve to determine the stationary values of b, T_L and the constant m. Eliminating m from Eqs. (18) and (19) gives

$$b^{1/2} = -\left[\frac{(\alpha - 1)^2 s A^2}{(\alpha + 1) k T_0^2}\right]^{1/6} \frac{[T_0^{\alpha+1} - 2(2\alpha + 1)(\alpha - 1)^{-1} T_L^{\alpha+1}]^{1/3}}{(T_0^{\alpha+1} - T_L^{\alpha+1})^{1/6}}. \tag{20}$$

Comparing Eqs. (16) and (20), it is seen that $t_0 > (\alpha - 1)/2(2\alpha + 1)$. Though Eqs. (17) and (20) are sufficient to determine the optimum value of b by eliminating T_L, it is more convenient to write Eq. (20) as

$$t_0 = \frac{1}{8(2\alpha + 1)} [4(\alpha - 1) - G + G^{1/2}\{24(\alpha + 1) + G\}^{1/2}], \tag{21}$$

where

$$G = (\alpha + 1) k b^3 / (2\alpha + 1) s A^2 T_0^{\alpha - 1} \tag{22}$$

and the positive sign in front of $G^{1/2}$ is chosen because $t_0 > (\alpha - 1)/2(2\alpha + 1)$. Equation (17) becomes

$$B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) = 2(\alpha + 1)(2\alpha + 1)^{-1/2} G^{-1/2} t_0^{\beta}, \tag{23}$$

where t_0 is given by Eq. (21). It is seen that the transcendental Equation (23) contains only the variables α and G. For a fixed $\alpha > 1$ and under the conditions $G > 0$ and $t_0 > (\alpha - 1)/2(2\alpha + 1)$, Eq. (23) is satisfied by one, and only one, value of G which, in turn, determines uniquely the optimum geometry of the fin from Eq. (22).

It remains to be shown that the extreme value of the function $q(b, T_L)$ does exist at the stationary point thus found and that the extreme value is a true maximum. For the existence of the extreme value, it is necessary that the two partial derivatives $\partial p/\partial b$ and $\partial p/\partial T_L$ shall not both vanish at the stationary point [5]. It can easily be shown that both derivatives will vanish only when $T_L = 0$ which is excluded in the present problem. The sufficient condition for the extreme value of q to be a maximum is that $d^2 q < 0$ at the stationary point. This is indeed true if $t_0 > (\alpha - 1)/2(2\alpha + 1)$.

5. Special case when $T_L = 0$. In the above discussion, the case when $T_L = 0$ is excluded. If $T_L = 0$, $C = 0$ from Eq. (7). A direct integration of Eq. (5) with boundary condition (2) gives

$$T^{-(\alpha-1)/2} - T_0^{-(\alpha-1)/2} = \left[\frac{(\alpha - 1)^2 s}{(\alpha + 1) k b}\right]^{1/2} x \tag{24}$$

$$q = \left(\frac{4skb}{\alpha + 1}\right)^{1/2} T_0^{(\alpha+1)/2}. \tag{25}$$

It is seen that no maximum value of q exists for any finite value of b. The non-existence of the extreme value is apparent since physically $T_L = 0$ requires the fin length $L \to \infty$.

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