ON COMPLETE SOLUTIONS FOR FRICTIONLESS EXTRUSION IN PLANE STRAIN*

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Consideration is given to the extension of partial slip line field upper bound solutions to give the true yield point load when the material is constrained. In particular, the problem of frictionless extrusion is studied and it is shown that it is possible to extend only one of three available partial solutions. If it is not possible to extend the available partial solution, the use of discontinuous stress fields leads to lower bound solutions, and examples of this technique are given.

1. Introduction. It is well known that all slip line field solutions constitute upper bound solutions, being partial incomplete solutions as described by Bishop (1953). This is because the slip line field is always associated with a kinematically admissible velocity field, although it is not generally associated with a statically admissible stress field, as discussed by Prager (1959, p. 118), for the extrusion problem. Bishop (1953) has proposed techniques for extending such partial stress solutions into neighbouring rigid regions, with a view to demonstrating whether or not a statically admissible stress field exists for the partial solution considered. If such a statically admissible stress field does exist, then the partial solution provides a lower bound also, giving in fact the true yield point load for the problem concerned. It is a consideration of the possibility of extending partial solutions to give true yield point loads for the problem of frictionless extrusion which forms the subject of this paper.

2. Available partial solutions, 3:1 ratio. For the 3:1thickness ratio of sheet extrusion through a square-edged die there exist three possible solutions, as shown in Fig. 1. All three solutions were proposed by Hill (1948) and in discussing the solutions shown in Fig. 1b and 1c he pointed out that the existence of dead metal regions would require the satisfying of certain conditions by the friction called into play on the surface of the dead metal.

Now the solution shown in Fig. 1(a) has the lowest yield point load, and since all three partial solutions are differing upper bound solutions, it should be possible to extend only that given by Fig. 1(a). This may not in fact be possible since there may be a further partial solution giving a still lower upper bound.

3. Complete solution, 3:1 ratio. A possible statically admissible stress field is shown in Fig. 2, for the partial solution of Fig. 1(a). Also shown in this figure is the stress plane showing the cycloidal traces of the pole of the Mohr's circle, using Prager's (1953) geometrical representation. The broken line 11 — 6" is a principal stress trajectory, and the stresses are transmitted across this boundary and supported on an infinite number of rectangular elements, via triangular elements of the type shown inset, the faces of the rectangular elements being normal and parallel to the extrusion axis. This concept was used by Bishop in the paper already referred to. The principal stress σ_p

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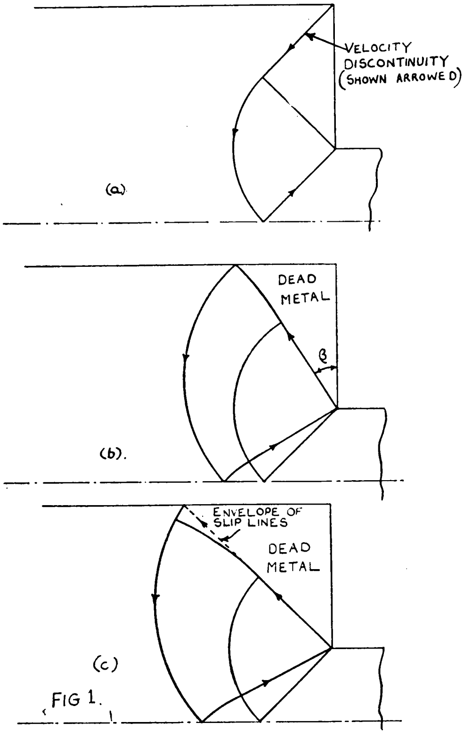
VELOCITY DISCONTINUITY (SHOWN ARROWED)
(a)
DEAD METAL
β
(b).
ENVELOPE OF SLIP LINES
DEAD METAL
(c)
FIG 1.

Fig. 1

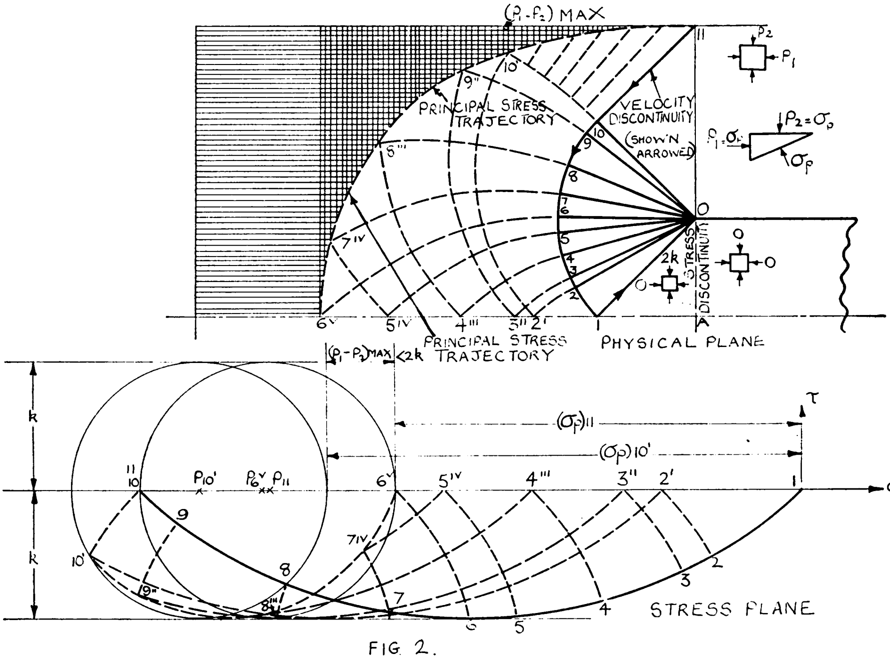
(P1-P2) MAX
PRINCIPAL STRESS TRAJECTORY
VELOCITY DISCONTINUITY (SHOWN ARROWED)
P2
P1
P2=σp
P1=σp
σp
STRESS DISCONTINUITY
2k
O
PRINCIPAL STRESS TRAJECTORY
PHYSICAL PLANE
(P1-P2)MAX <2k
(σp)11
(σp)10'
τ
σ
k
k
STRESS PLANE
FIG 2.

Fig. 2

on any triangular element can be supported by two direct stresses p_1 and p_2 both equal to σ_p, so that no shear stresses are invoked. The rigid material to the left of the principal stress trajectory will not yield provided that $| p_1 - p_2 | < 2k$. The maximum value of this stress difference occurs for the shaded element and is of amount $(\sigma_p)_{11} - (\sigma_p)_{10'}$ where $(\sigma_p)_{11}$ and $(\sigma_p)_{10'}$ are the principal stresses acting normal to the principal stress trajectory at the points 11 and 10′ respectively. The broken line $0A$ is a stress discontinuity, as suggested by Bishop, the stress states on each side of it being as indicated in the diagram. Since the infinity of columns transmit no shear stress, the boundary, equilibrium, and yield conditions are everywhere satisfied and the solution is a statically admissible stress field constituting a lower bound. Since it is also an upper bound, the actual yield point load will be associated with it.

Considering the solution shown in Fig. 1(b), it can be shown immediately that an associated statically admissible stress field does not exist, as follows. The angle β is less than $\pi/4$, and since the shear stress on the vertical die face must be made zero to comply with the requirements of the stress boundary conditions, the rigid dead metal vertex at 0 cannot support the boundary shear stresses postulated without yielding. That this is so can be seen from the analysis by Hill (1954) (case iv), in which it is proved that the angle β cannot be less than $\pi/4$.

Considering the solution shown in Fig. 1(c), an attempt to extend the partial solution is shown in Fig. 3. On the left hand side of the principal stress trajectory $10' - 6^{v}$, the maximum stress difference (in the rectangular element A) is less than $2k$. On the right hand side of the envelope of slip lines $9 - 10'$, an equilibrium stress state on each triangular element can be as shown in the inset diagram, from which it is seen that

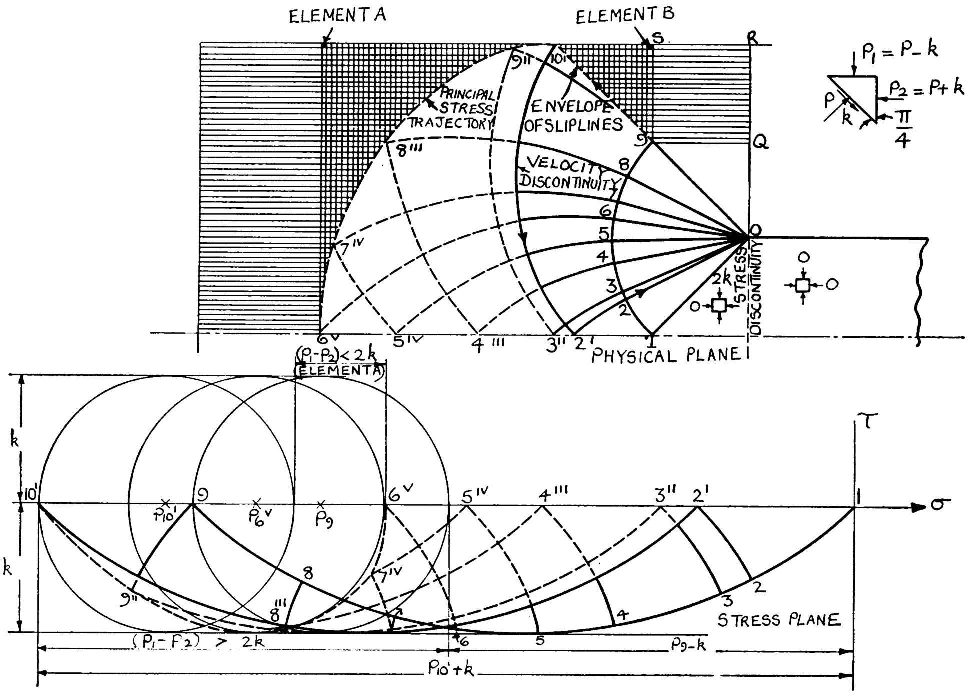

FIG. 3

$p_1 = p - k$, $p_2 = p + k$, where p is the mean stress at any point on the envelope. Thus at the point 9, $p_1 = p_9 - k$, and at the point 10′, $p_2 = p_{10'} + k$, as shown in the stress plane. Since the Mohr's circles for these two points are certainly not coincident, the element B is greatly overstressed in that $p_1 - p_2 \gg 2k$. This is true for the whole of the region $10' - R - Q - 9$ of the rigid dead metal, so that this region would yield under the distribution of stress applied to it. It is interesting to note that Hill's (1954) analysis is not applicable to the whole of the vertex $9 - 10' - S$, since the problem of the *variation* of stress on the face of such a vertex is not there considered. This extension of Bishop's infinity of rectangular elements permits the consideration of the possible yielding of such assumed rigid domains.

4. Extrusion ratios greater than 3:1. In Fig. 4 is shown the extended partial solution giving the true yield point load for extrusion ratios greater than 3:1, together with the stress plane, using the methods just described. The complete range of validity of this solution has not been investigated.

5. Other extrusion ratios. In Fig. 5 is shown a statically admissible stress field for an extrusion ratio of 2:1. This solution is interesting in that it has a dead metal zone, even with a frictionless interface between container and material. There is a field of constant stress in the dead metal region, the stress state being as shown in the inset element. It has not been found possible as yet to find complete solutions for other ratios.

6. Discussion. It has been demonstrated that the techniques suggested by Bishop

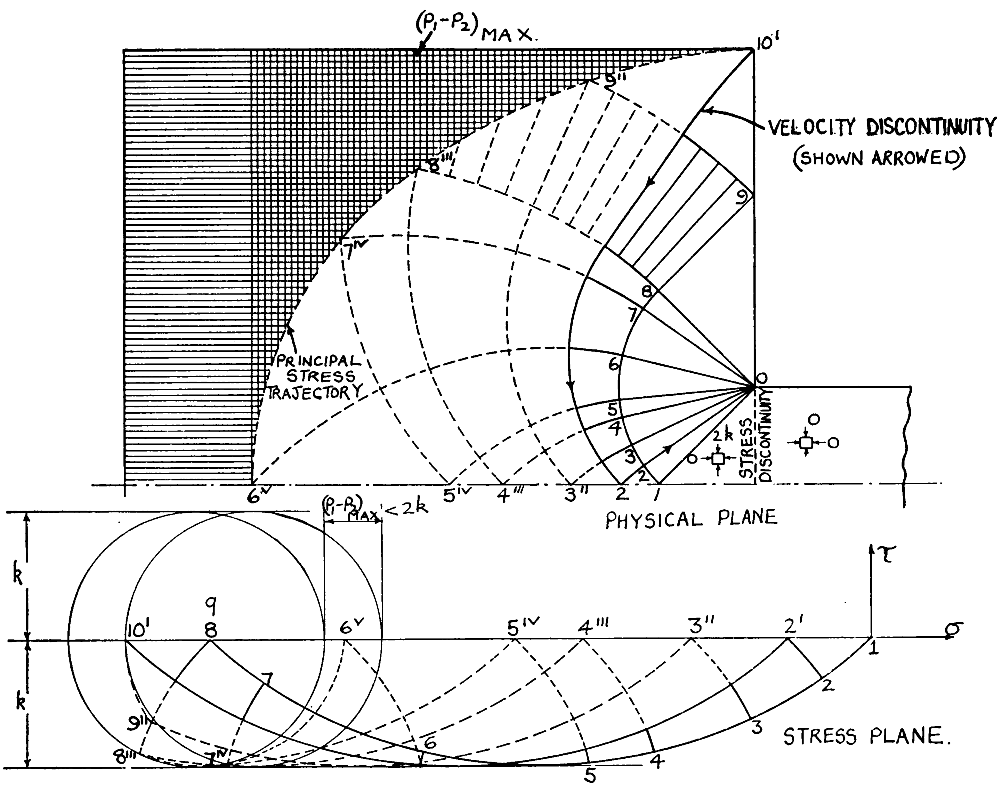

FIG. 4

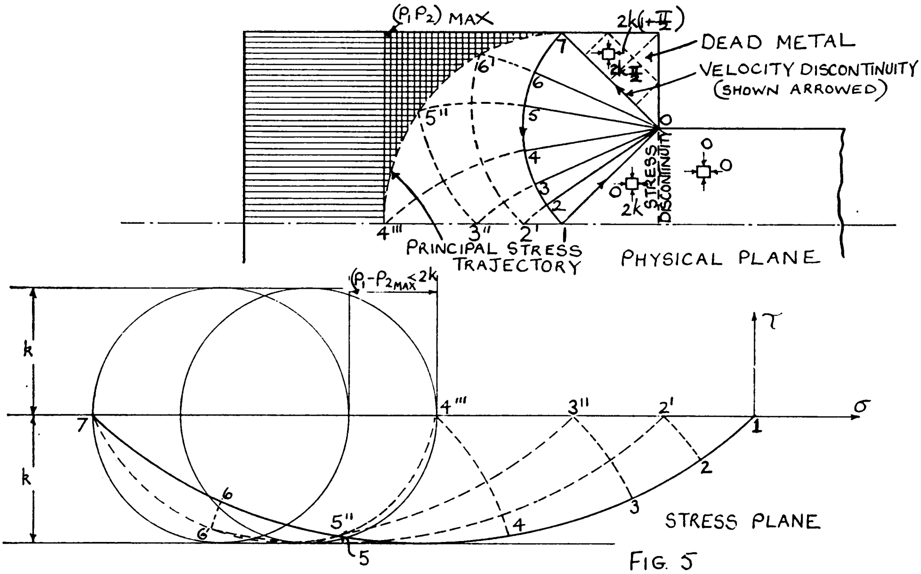

Fig. 5

for extending partial solutions can be applied to problems of contained deformation. In this connection, it is the author's opinion that Bishop's extension of Hill's field suggested in his paper (Bishop, 1953) is unacceptable because it violates the stress boundary conditions at the frictionless container—material interface. Problems of contained deformation with friction at such interfaces require special consideration, particularly concerning whether or not relative motion exists at such boundaries. The formulation of lower bound theorems for such cases constitutes a difficult problem.

It may often happen that *none* of the available partial solutions can be extended to give a statically admissible stress field, and hence the true solution. Under these circumstances a field of stress discontinuities in statical equilibrium may be found which would give a lower bound solution.

As examples, two discontinuous stress fields for the 3:1 ratio frictionless extrusion problem are shown in Figs. 6 and 7, in which the broken lines are the stress discontinuities separating regions of constant stress. The magnitudes of the constant stresses in each region are shown by the inset elements whose sides are parallel with the principal planes in the particular region concerned. In the stress plane of each diagram are shown the Mohr's circles for the states of stress in each of the regions, together with their 'poles', as defined by Prager (1953). The slip lines of both these fields intersect all boundaries and the axis of symmetry at angle $\pi/4$, so that they are statically admissible, and therefore lower bound solutions. The field shown in Fig. 6 leads to a mean extrusion pressure of $8/3k$, whilst that of Fig. 7 gives $10/3k$. (The field shown in Fig. 7 was derived from that suggested by Shield and Drucker (1953) for the indentation problem).

For this particular problem it has been possible to show that the upper bound slip line field of Fig. 2 gives, in fact, the true yield point load of the deformation, namely a mean extrusion pressure of $2k/3\ (2 + \pi) \simeq 10.284/3k$. Had this not been known, however, taking the mean extrusion pressure midway between the best available upper and lower bounds would have resulted in an error of about $1\frac{1}{2}\%$ only.

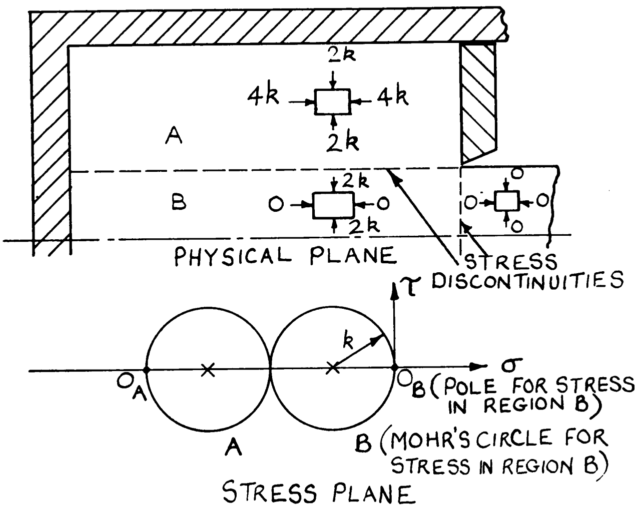
2k
4k
4k
A
2k
2k
B
O
O
2k
O
O
O
O
PHYSICAL PLANE
STRESS DISCONTINUITIES
τ
k
σ
O_A
O_B (POLE FOR STRESS IN REGION B)
A
B (MOHR'S CIRCLE FOR STRESS IN REGION B)
STRESS PLANE

FIG. 6

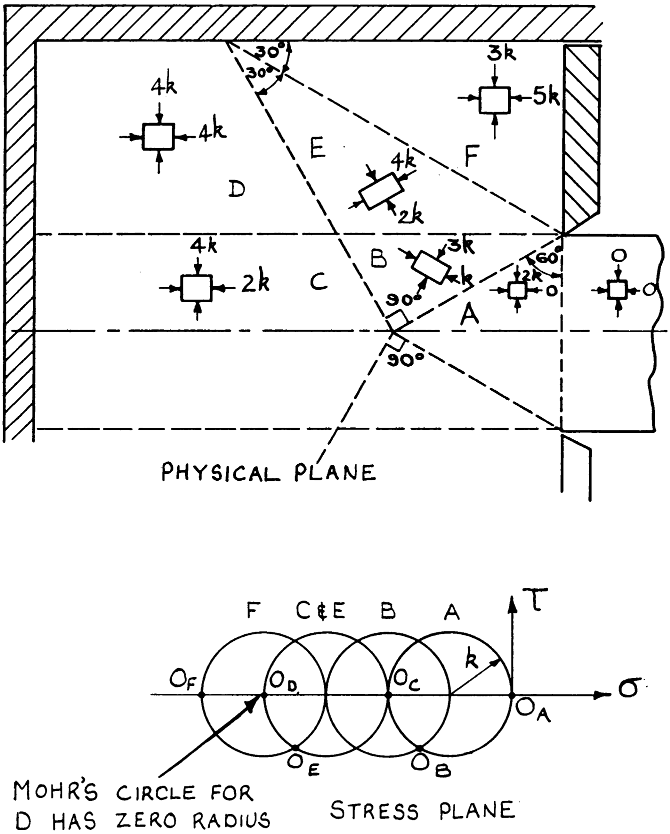
30°
30°
3k
5k
4k
4k
E
F
4k
2k
D
4k
3k
2k
B
k
60°
2k
O
O
O
C
90°
A
90°
PHYSICAL PLANE
F
C & E
B
A
τ
k
O_F
O_D
O_C
O_A
σ
O_E
O_B
MOHR'S CIRCLE FOR D HAS ZERO RADIUS
STRESS PLANE

FIG. 7

Upper bound solutions for many problems of metal working have been proposed by Johnson (1959); the lower bound techniques proposed here form a complementary method of attacking such problems. The possibility of extending the upper bound methods to deal with three dimensional problems has been suggested previously, Alexander (1959); the use of discontinuous stress fields has been shown by Shield and Drucker (1953) to be suitable for application to three dimensional cases for finding lower bounds. Thus methods are available for bounding yield point loads for three-dimensional metal-working problems, which is a field requiring further study.

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