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A NOTE ON AN INVERSE PROBLEM IN MATHEMATICAL PHYSICS*

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1. Introduction. The problem of solving the linear partial differential equation of diffusion theory and heat conduction,

$$\varphi(x) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

subject to appropriate initial and boundary conditions has been extensively investigated using a variety of analytic techniques. We wish to consider a type of problem associated with this equation which has not been investigated to any extent, the inverse problem.

In place of assuming that the coefficient function $\varphi(x)$ is known and trying to find the solution $u(x, t)$, we shall suppose that the solution is known at certain points of the x, t plane, and try to determine the function $\varphi(x)$ from this information. This type of question appears first to have been asked by the mathematical physicist Ambarzumian, and it was also posed by Langer, who gave an analytic procedure for determining the coefficient function. The first detailed analysis of this problem was given by Borg [1], where references to earlier work by Langer and others may be found. Similar questions arise in the quantum theory of scattering, and were, in principle, solved by Gelfand and Levitan [2]. The most recent paper is that of Kay [3], where references to work of Levinson and others may be found.

The aim of this paper is to show how problems of this nature are closely related to some classical synthesis problems for electrical networks. Our new approach furnishes a way of obtaining approximate solutions which appears to be more useful from the computational, engineering and physical point of view than the exact solutions furnished implicitly by the methods of Borg and Gelfand and Levitan.

2. The analytic problem. Consider the equation of (1.1) taken to hold for $0 < x < L$, $t > 0$, subject to the initial and boundary conditions

$$\begin{aligned} &\text{(a)} \quad u(L, t) = 0, \\ &\text{(b)} \quad u(x, 0) = 0, \\ &\text{(c)} \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left[Z\left(\frac{d}{dt}\right)u(x, t)\right]_{x=0}. \end{aligned} \tag{1}$$

Here $Z(s)$ is a meromorphic function of s which we shall specify in further detail below. Taking Laplace transforms with respect to t, we obtain the Sturm-Liouville problem

$$\begin{aligned} &\text{(a)} \quad s\varphi(x)v = \frac{d^2v}{dx^2}, \\ &\text{(b)} \quad v(L, s) = 0, \\ &\text{(c)} \quad \left.\frac{dv}{dx}\right|_{x=0} = Z(s)v\Big|_{x=0}, \end{aligned} \tag{2}$$

*Received February 8, 1961.

where

$$v(x, s) = \int_0^\infty u(x, t)e^{-st}\, dt. \tag{3}$$

Given the function $Z(s)$, we wish to determine the density function. For an appropriate choice of $Z(s)$, this problem is equivalent to the problem considered by Borg. To see this, suppose that $Z(s)$ has the form

$$Z(s) = \prod_{n=1}^{\infty} (1 + s/\lambda_n) / \prod_{n=1}^{\infty} (1 + s/\mu_n), \tag{4}$$

where λ_n , $\mu_n > 0$. If $s = -\lambda_n$, we have the Sturm-Liouville problem

$$\begin{aligned} v'' + \lambda_n v &= 0, \\ v'(0) = v(L) &= 0. \end{aligned} \tag{5}$$

If $s = -\mu_n$, we have the problem

$$\begin{aligned} v'' + \mu_n v &= 0, \\ v(0) = v(L) &= 0. \end{aligned} \tag{6}$$

We thus have the two distinct problems required for the solution of our problem following Borg's methods. Unfortunately, Borg's results do not furnish any useful technique for actually obtaining the unknown function, $\varphi(x)$. We shall present another approach which thus furnishes an alternative attack upon Borg's problem.

3. Discrete approximation. Let us, following a standard route, replace the differential equation of (2.2a) by the difference equation

$$v_{k+1} - 2v_k + v_{k-1} = \Delta^2 s\varphi_k v_k , \qquad k = 1, 2, \cdots , N - 1, \tag{1}$$

where

$$v_k = v(k\ \Delta), \varphi_k = \varphi(k\ \Delta), \qquad N\ \Delta = L. \tag{2}$$

The boundary conditions of (2.2b) and (2.2c) are replaced by

$$\begin{aligned} &\text{(a)} \qquad & v_N &= 0, \\ &\text{(b)} \qquad & v_1 - v_0 &= \Delta Z_N(s) v_0 . \end{aligned} \tag{3}$$

We replace the meromorphic function $Z(s)$ by its approximation

$$Z_N(s) = \frac{\prod_{n=1}^{N} (1 + s/\lambda_n)}{\prod_{n=1}^{N} (1 + s/\mu_n)}. \tag{4}$$

4. Network theory. Once the problem has been transformed into that of determining the values of φ_k , an approximation to the determination of $\varphi(x)$, we see that it is equivalent to that of determining the elements of a two-port network, a network of transmission-line type, given the response function.

Probably the most efficient and convenient way of deriving the values of the φ_k is based upon the use of continued fraction expansions. The details of this procedure, together with various necessary and sufficient conditions upon $Z(s)$ and the zeros and poles, $-\lambda_n$ and $-\mu_n$, may be found in Guillemin [4], Weinberg [5], Weinberg and Slepian [6], where many further references may be found.

Consider the system of equations

$$v_{k+1} - (2 - \Delta^2 s\varphi_k)v_k + v_{k-1} = 0, \qquad k = 1, 2, \cdots, N - 1. \tag{1}$$

Writing

$$\frac{v_{k-1}}{v_k} = (2 - \Delta^2 s\varphi_k) - 1/(v_k/v_{k+1}), \tag{2}$$

we see that we obtain the finite continued fraction

$$\frac{v_0}{v_1} = (2 - \Delta^2 s\varphi_1) - \cfrac{1}{(2 - \Delta^2 s\varphi_2) - \ddots} \tag{3}$$

Hence, any method which permits us to write v_0/v_1, as given by (3.3b), as a continued fraction of this form, yields the values $\varphi_1, \varphi_2, \cdots, \varphi_N$, and thus gives us a computational solution of the discrete inverse problem.

Questions of degree of approximation will be discussed subsequently.

That there is a strong connection between the mathematical problems of network theory and those of quantum mechanics is no longer surprising in view of the work of Wigner and others on positive real functions. See Wigner [7], Lane and Thomas [8].

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