ON YIELD CONDITIONS IN GENERALIZED STRESSES*

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1. Introduction. Generalized stresses and strain rates were introduced in limit analysis by Prager [1]. They were defined as the stress-type variables that appear in the expression of the power of dissipation, the stress variables being components of the stress tensor, stress resultants or dimensionless stress resultants. It was then emphasized by Prager [2] that other stress-type variables exist, though not entering in the expression of the power of dissipation because they are related to zero strain rates. These stress variables he called "reactions," to distinguish them from those which he called "generalized stresses." Numerous applications of these concepts have been made up to now; references to most of these can be found in Prager [2] and Hodge [3]. These applications essentially concern plate or shell problems, where the generalized stresses are the stress resultants and bending or twisting moments.

The question arises whether it is always possible to express the yield condition in terms of stress variables entering in the expression of the power of dissipation, i.e., in terms of the so-called generalized stresses *only*. Physically, the possibility of doing this is not obvious because the yield condition essentially involves the stresses, and one might think that the non-zero "reactions," because they produce in general non-zero stresses, would enter the yield condition.

We shall show first that it is indeed always possible to express the yield condition in terms of the generalized stresses only and, moreover, that the elimination of the "reactions" can be achieved automatically, by completely ignoring these "reactions."

Then, considering the case where one intends to obtain such a yield condition by purely statical considerations, we shall prove a theorem that enables one to eliminate the reactions from the yield condition in an easy and direct manner.

Finally, we shall illustrate the theorem by two examples and present some short concluding remarks.

2. Yield conditions in generalized stresses. Let us consider stress variables $S_1, S_2, \cdots S_n$ used to describe the state of stress of a rigid-perfectly plastic continuum. In general, the yield condition in its normalised form, is

$$F(S_1, S_2, \cdots S_n) = 1; \tag{1}$$

it is represented by a certain hypersurface in the space with the rectangular cartesian coordinates $S_1, S_2, \cdots S_n$.

Let us denote $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n$ the corresponding strain rates, and suppose that we have

$$\dot{q}_i = 0 \qquad (i = k + 1, \cdots n). \tag{2}$$

In this case, S_{k+1} to S_n are reactions, while S_1 to S_k are generalized stresses.

Let the rectangular axes $\dot{q}_i$ coincide with the axes S_i $(i = 1, \cdots n)$. If Drucker's quasi-thermodynamic postulate [4] and its consequences are accepted, conditions (2) require that the stress point be on the yield surface (1) at regular points where the

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normal is perpendicular to all S_{k+1} to S_n axes or at singular points where this direction is on or inside the cone of the outside pointing normals at neighbouring points.

The points at which conditions (2) are satisfied form a certain hypersurface that, projected into the subspace $S_1, S_2, \cdots S_k$, gives a yield condition of the form

$$\Phi(S_1, S_2, \cdots S_k) = 1. \tag{3}$$

This projection is always possible.

In fact, the only case of impossibility would occur if at least one point determined by (2) on the yield surface (1) would be on one of the S_{k+1} to S_n axes, say S_l. The projection which, strictly speaking would still be possible, would give, instead of a certain set of $S_1, S_2, \cdots S_k$ satisfying (3), the value $S_1 = S_2 = \cdots = S_k = 0$.

But this case is impossible because, due to the convexity of the yield surface, this would put the origin on or outside the yield surface, which is not admissible.

This being shown, we can call "generalized stresses" the stress variables entering in the expression of the power of dissipation or in the yield condition, denote them by $Q_1, Q_2, \cdots Q_k$ and re-write (3) as

$$\Phi(Q_1, Q_2, \cdots Q_k) = 1. \tag{3'}$$

The yield condition (3) can be obtained directly, ignoring completely the "reactions," as it was done in [5] and elsewhere.

To show this in general terms, let us return to (1) and consider the power of dissipation

$$D = S_1 \cdot \dot{q}_1 + S_2 \cdot \dot{q}_2 + \cdots + S_n \cdot \dot{q}_n . \tag{4}$$

The perfectly plastic continuum being inviscid, D is an homogeneous function of order one of $\dot{q}_1, \cdots \dot{q}_n$ and we can write

$$D = \frac{\partial D}{\partial \dot{q}_1} \cdot \dot{q}_1 + \cdots + \frac{\partial D}{\partial \dot{q}_n} \cdot \dot{q}_n . \tag{5}$$

Comparing (4) to (5) we obtain

$$S_i = \frac{\partial D}{\partial \dot{q}_i} \qquad (i = 1, 2, \cdots n). \tag{6}$$

Equations (6) are a parametric definition of (1), the parameters being the $\dot{q}_i$ (or their ratios).

Now if we have conditions (2), (4) reduces to

$$D^* = Q_1 \cdot \dot{q}_1 + Q_2 \cdot \dot{q}_2 + \cdots + Q_k \cdot \dot{q}_k ,$$

the stress variables S_1 to S_k being generalized stresses which we denote consequently Q_1 to Q_k. The yield condition (3') is given by the first k equations (6) where conditions (2) are entered.

But, as only *partial* differentiation is used, we obtain the same relation in the $\dot{q}_i$ $(i = 1, \cdots k)$ if we form $\partial D/\partial \dot{q}_i$ $(i = 1, \cdots k)$ and use $\dot{q}_i = 0$ $(i = k + 1, \cdots n)$ or if we form $\partial D^*/\partial \dot{q}_i$.

Thus, the first k equations (6) are identical to

$$S_i = \frac{\partial D^*}{\partial \dot{q}_i} \qquad (i = 1, \cdots k). \tag{7}$$

Equations (7) give directly (3′) under parametric form, ignoring completely all reactions S_{k+1} to S_k .

We shall now discuss the following two points:

1. Does the normality law apply to the yield condition (3′) and, more generally, in arbitrary linear subspaces?
2. Is it still correct to ignore "reactions" when the yield condition is established by purely statical considerations or, in other words, how can reactions be eliminated from (1)?

3. Normality law in subspaces. The validity of the normality law in general subspaces is by no means obvious and should be investigated first.

We denote by $S_1, \cdots S_n$ all non-vanishing stress variables. The yield condition in its normalised form is (1) where all non-vanishing stress variables enter, in general.

In the most general way, one obtains a certain simplified yield condition in a certain subspace when there exist a certain number of relations between the stresses, say $(n - k)$ relations,

$$\begin{cases} f_1(S_1, \cdots S_n) = 0 \\ f_2(S_1, \cdots S_n) = 0 \\ \vdots \\ f_{n-k}(S_1, \cdots S_n) = 0. \end{cases} \tag{8}$$

If we suppose that we are able to express, by means of (8), $(n - k)$ stresses as functions of the others, Eqs. (8) can be re-written

$$\begin{cases} S_{k+1} = S_{k+1}(S_1, S_2, \cdots S_k) \\ S_{k+2} = S_{k+2}(S_1, S_2, \cdots S_k) \\ \vdots \\ S_n = S_n(S_1, S_2, \cdots S_k). \end{cases} \tag{9}$$

Introduction of (9) in (1) furnishes a simplified yield condition

$$\Psi(S_1, \cdots S_k) = 1 \tag{10}$$

in the subspace $(S_1, \cdots S_k)$.

We consider first regular yield surfaces.

The normality law, applied to the initial yield condition (1) gives the strain rates

$$\dot{q}_i = \lambda \frac{\partial F}{\partial S_i} \qquad (i = 1, 2, \cdots n), \tag{11}$$

λ being a positive constant.

The normality law remains valid for the yield condition (10) in the subspace $(S_1, \cdots S_k)$ if we can write*

$$\dot{q}_i = \lambda \frac{\partial \Psi}{\partial S_i} \qquad (i = 1, \cdots k). \tag{12}$$

*$\dot{q}_i$ for $i = k + 1, \cdots n$ cannot be obtained of course.

We therefore have

$$\frac{\partial F}{\partial S_i} = \frac{\partial \Psi}{\partial S_i} \qquad (i = 1, \cdots k). \tag{13}$$

On the other hand, we know that

$$\Psi(S_1, \cdots S_k) \equiv F[S_1, \cdots S_k, S_{k+1}(S_1, \cdots S_k), \cdots S_n(S_1, \cdots S_k)] \tag{14}$$

Applying chain differentiation, we find

$$\frac{\partial \Psi}{\partial S_i} = \frac{\partial F}{\partial S_i} + \frac{\partial F}{\partial S_{k+1}} \cdot \frac{\partial S_{k+1}}{\partial S_i} + \cdots + \frac{\partial F}{\partial S_n} \cdot \frac{\partial S_n}{\partial S_i} \qquad (i = 1, \cdots k). \tag{15}$$

In general we cannot expect relations (8), or their explicit form (9), to be such that (15) reduces to (13) and so "in general, the normality law is not applicable to simplified yield conditions."

This law will however continue to be valid in two important cases:

a) If relations (9) have the special form

$$S_i = K_i \qquad (i = k + 1, \cdots n), \tag{16}$$

where the K_i are constants (such, of course, that the yield condition is not violated).

In this case we have

$$\frac{\partial S_{k+1}}{\partial S_i} = \frac{\partial S_{k+2}}{\partial S_i} \equiv \cdots = \frac{\partial S_n}{\partial S_i} = 0, \tag{17}$$

and (15) reduces to (13).

b) If S_{k+1} to S_n are "reactions."

In this case we have

$$q_i = 0 \qquad (i = k + 1, \cdots n) \tag{18}$$

and, applying the normality law (11), relations (8) become

$$\frac{\partial F}{\partial S_i} = 0 \qquad (i = k + 1, \cdots n) \tag{19}$$

or, explicitly,

$$\frac{\partial F}{\partial S_{k+1}} = \frac{\partial F}{\partial S_{k+2}} = \frac{\partial F}{\partial S_n} = 0 \tag{20}$$

and (15) reduces once again to (13).

We may sum up our findings as follows.

"The normality law remains applicable to simplified yield conditions obtained by giving constant values to a certain number of stress variables or by eliminating the reactions."

Let us now turn to yield surfaces with singular points. The normality law is generalized by the condition that the strain rate vector (q_i) be in the cone of the outside pointing normals at the neighbouring points of the pointed vertex.

Equation (10) represents the cylinder with generating lines parallel to the axes of S_{k+1} to S_n, projecting the intersection of (1) with the hypersurfaces (8). In the space $S_1, \cdots S_k$, it is the equation of the cross section of this cylinder.

The normality law will remain valid for the hypersurface (10) if, at a pointed vertex P of (1) that is also on (8), the projection of the cone of normals at (1) in P is the cone of normals at (10) at the point P' projection of P.

We consider now a pointed vertex as the limit of a very small region where the direction of the normal varies very quickly but continuously.

Suppose that this region is bounded on the yield surface by a certain curve C, whose limit is the point P at the vertex. When C tends to P, we can at each stage express Eq. (13) in the following terms: the projection of the cone of normals along C are the normals to the projection of C.

Considering the conditions of reduction of (15) to (13) and passing to the limit, we readily extend our preceding conclusions to singular yield surfaces.

4. Theorem on the adaptation of the reactions. Let us now suppose that, among the totality of n non-vanishing stress variables, the k first ones are generalized stresses $Q_1, Q_2, \cdots Q_k$ and the others are reactions $S_{k+1}, S_{k+2}, \cdots S_n$.

We give fixed values to all generalized stresses but one, say Q_k

$$\begin{cases} Q_1 = K_1 \\ Q_2 = K_2 \\ \vdots \\ Q_{k-1} = K_{k-1}. \end{cases} \tag{21}$$

We obtain a simplified yield condition

$$\varphi(Q_k, S_{k+1}, \cdots S_n) \equiv F(K_1, K_2, \cdots K_{k-1}, Q_k, S_{k+1}, \cdots S_n) = 1 \tag{22}$$

when we use (21) in (1).

Suppose that the yield surface is regular.

The normality law being valid for the yield condition (22), we have

$$q_i^{\cdot} = \lambda \frac{\partial \varphi}{\partial S_i} \qquad (i = k + 1, \cdots n), \tag{23}$$

(and also $q_k^{\cdot} = \partial\varphi/\partial Q_k$ which is of no interest for our purpose).

The stress variables S_{k+1} to S_n being reactions,

$$q_i^{\cdot} = 0 \qquad (i = k + 1, \cdots n). \tag{24}$$

Comparing (23) and (24) we obtain,

$$\frac{\partial \varphi}{\partial S_i} = 0 \qquad (i = k + 1, \cdots n), \tag{25}$$

because $\lambda \neq 0$.

Equations (25) are extremum conditions for Q_k, considered as a function of $S_{k+1}, \cdots S_n$ implicitly defined by (22).

Let us note, that, as the yield surface is convex at *all* points, relations (25) are conditions of absolute maximum or minimum.

We can then state the *adaptation theorem*:

"If we fix all generalized stresses but one, the reactions adapt themselves in such a way as to give to the non-fixed generalized stress a maximum positive value or a minimum negative value."

The theorem remains valid for a singular yield surface as can be readily shown: the generalized normality law applies to the yield surface (22), and conditions (24) determine those points on (22) where the strain rate vector is parallel to the Q_k axis. If this occurs at a pointed vertex, Q_k assumes an extreme value at this point because of the convexity of the yield surface, which contains the origin.

5. Examples of application. We consider a rigid-perfectly plastic cylindrical shell without axial load, under axially symmetrical loading. As in most plate and shell problems, we suppose that straight lines normal to the undeformed middle surface remain normal to the deformed surface during yielding. The axis $0Z$ directed along the normal at a typical point of the middle surface of the shell (Fig. 1) is then a principal axis of

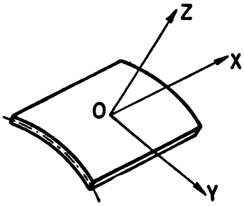

FIG. 1

the strain rate tensor. Consequently, as Ziegler [6] has shown, in the case of the Tresca-Guest yield condition as well as in the case of the von Mises yield condition, it is also a principal axis of the stress tensor and $\tau_{zy} = \tau_{zx} = 0$ at all levels in the thickness of the shell. Consequently, to be consistent with the previous hypothesis, shearing forces normal to the middle surface in a completely plastic element of the shell (or of the plate) must vanish exactly. Moreover, we suppose $\sigma_z = 0$. We are then exactly in plane stress conditions.

Due to the symmetry of our problem, the only non-vanishing stress variables are M_x , M_θ and N_θ . (Fig. 2)

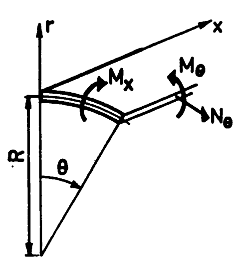

FIG. 2

Symmetry also imposes the condition that the circumferential rate of curvature $\dot{\chi}_\theta$ be zero.

M_θ being then a reaction, we have to determine the yield condition in M_x and N_θ .

1. Let us first consider a sandwich shell, obeying von Mises' yield condition. This shell is composed of two thin sheets, each of the thickness $t/2$, which are separated by a core of the thickness h. (Fig. 3). The thin sheets support only normal stresses, uniformly distributed on the thickness.

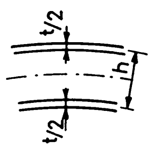

FIG. 3

If we call

$\sigma_{\theta t}$ and σ_{xt} the stresses in the top sheet
$\sigma_{\theta b}$ and σ_{xb} the stresses in the bottom sheet

we have

$$\begin{cases} N_\theta = \dfrac{t}{2}\,(\sigma_{\theta t} + \sigma_{\theta b}), \\[2ex] M_\theta = \dfrac{th}{4}\,(\sigma_{\theta t} - \sigma_{\theta b}), \\[2ex] M_\theta = \dfrac{th}{4}\,(\sigma_{xt} - \sigma_{xb}). \end{cases} \tag{26}$$

Introducing dimensionless stress variables

$$n_\theta = \frac{N_\theta}{N_0}\,; \qquad m_\theta = \frac{M_\theta}{M_0}\,; \qquad m_x = \frac{M_x}{M_0}\,,$$

where $N_0 = \sigma_0 \cdot t$ and $M_0 = \sigma_0 \cdot th/2$,

σ_0 being the yield stress of the sheets in pure tension,

we get

$$\begin{cases} n_\theta = \dfrac{\sigma_{\theta t} + \sigma_{\theta b}}{2\sigma_0}\,, \\[2ex] m_\theta = \dfrac{\sigma_{\theta t} - \sigma_{\theta b}}{2\sigma_0}\,, \\[2ex] m_x = \dfrac{\sigma_{xt} - \sigma_{xb}}{2\sigma_0}. \end{cases} \tag{27}$$

von Mises' yield condition

$$\sigma_x^2 + \sigma_\theta^2 - \sigma_x\sigma_\theta = \sigma_0^2 \tag{28}$$

is shown in Fig. 4 in the $(0\sigma_x\,,\, 0\sigma_\theta)$ plane. In this plane, the state of stress in the shell is represented by a straight segment tb, coordinates of t and b being respectively $(\sigma_{\theta t}\,,\, \sigma_{xt})$ and $(\sigma_{\theta b}\,,\, \sigma_{xb})$. In view of (27), coordinates of the center c of tb give n_θ (and n_x) and projection of the segment tb on the axes give m_x and m_θ (the positive factors $1/\sigma_0$ and $1/2\sigma_0$ do not create any trouble).

As $n_x = 0$, the point c must stay on the $0\sigma_\theta$ axis. For a given position of this point (between A and B), corresponding to a given value of n_θ, plasticity of the shell element requires one point t or b at least to be on the yield locus.

The adaptation theorem tells us now that the inclination of tb must be taken so that its projection on $0\sigma_x$ be a maximum. This readily gives the condition

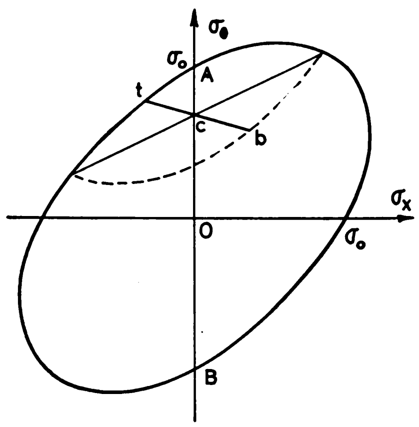

FIG. 4

$$m_\theta = \frac{m_x}{2} \tag{29}$$

which moreover puts both points, t and b, on the yield locus. Displacing the point c from A to B and respecting condition (29), we obtain without difficulty and directly, the elliptic relation

$$\tfrac{3}{4}m_x^2 + n_\theta^2 = 1 \tag{30}$$

between m_x and n_θ .

2. Let us consider the same shell, but obeying now Tresca-Guest's yield condition. A very similar analysis, with the use of the adaptation theorem gives the following results (Fig. 5)

a. when $0 \leq n_\theta \leq \frac{1}{2}$,

$$n_\theta \leq \frac{m_\theta}{m_x} \leq 1 - n_\theta \quad \text{and} \quad m_x = 1 \tag{31}$$

b. when $\frac{1}{2} \leq n_\theta \leq 1$, $m_\theta = m_x/2$ and

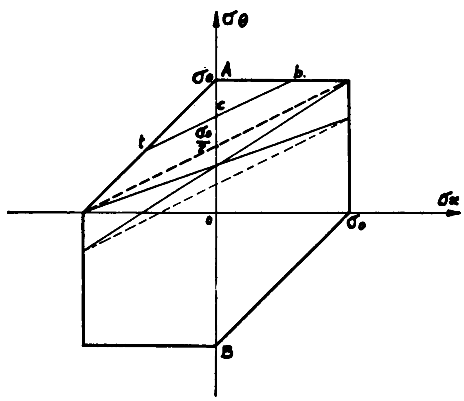

FIG. 5

$$\frac{m_x}{2} + n_\theta = 1 \tag{32}$$

The yield condition is given by (31) and (32).

6. Concluding remarks. As has been shown in Sec. 3, the adaptation theorem is by no means the only way to obtain the correct yield condition in generalized stresses. Even in the case of a purely statical procedure, reactions can also be eliminated by other considerations.

It should be pointed out, however, that the maximizing procedure applied in Sec. 5 has already been used (see [7] as an example).

The method was open to criticism at that time because it was not certain that reactions, which were ignored, would in fact take values which permitted this maximizing.

Our theorem gives, *a posteriori*, a rigorous justification of this method.

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