A VARIATIONAL PROBLEM RELATING TO COOLING FINS WITH HEAT GENERATION*

BY

CHEN-YA LIU

Carnegie Institute of Technology, Pittsburgh, Pa.

1. Introduction. Cooling fins are used in heat exchange apparatus to increase the rate of heat transfer. To economize in the design, we wish to know the shape of the fin which gives the maximum dissipation of heat for a given weight of the fin. For pure conduction fins, a criterion for this optimum fin problem was proposed by Schmidt [1] and recently proved by Duffin [2]. Cooling fins are also used in atomic reactors where heat is produced inside the fin as a result of atomic reaction. The question immediately arises: What is the optimum fin geometry in such cases? The answer to this question becomes more important in view of the industrial trend of trying to develop airborne reactors where the weight limitation is the most significant problem.

In the first part of this paper, the problem of cooling fins with heat generation is recast in a form suitable for treatment by the calculus of variations. The heat generation function which is not clearly known in our present state of knowledge is assumed as a function of the coordinate along the fin. The relation of the temperature to the heat generation is assumed to be linear. Euler equations are obtained by formal variational methods. Contrary to the case for pure conduction fins, the equations are not linear. General solutions to the Euler equations cannot be obtained in explicit form. However, sufficient conditions are derived for solving this optimum fin problem.

The second part of this paper concerns the solution when the heat generation function is linearly dependent on the temperature only. The temperature is found to be a hyperbolic sine function. This result is used to derive explicit expressions for the fin shape and for the maximum heat dissipation, for the cases of a rectangular fin and a circular fin.

2. The variational problem. Under the assumption that the heat generated is linearly proportional to the temperature u, the governing equation for a cooling fin can be written as

$$\frac{d}{dx}\left[y(x)\frac{du}{dx}\right] = [p(x) - q(x)y(x)]u, \tag{1}$$

where the known functions $p(x)$ and $q(x)$ representing the surface convection and heat generation effects respectively are positive and continuous.

The function $y(x)$ related to the fin shape satisfies the conditions

$$y(x) \text{ is differentiable} \tag{2}$$

$$y(x) > 0 \quad \text{for} \quad 0 \leq x < b \tag{3}$$

and

$$\int_0^b y(x)\,dx = K, \quad \text{a given constant (fin volume)}, \tag{4}$$

where b is the fin length.

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Also u is to satisfy the two boundary conditions

$$u = 1 \quad \text{at} \quad x = 0 \tag{5}$$

$$y(x)\,\frac{du}{dx} = 0 \quad \text{at} \quad x = b. \tag{6}$$

Under these conditions, we seek two functions $u(x)$ and $y(x)$ and the constant number b such that the cooling effect of the fin H defined as

$$H = \int_0^b [p(x) - q(x)y(x)]u(x)\,dx \tag{7}$$

gives a maximum.

Further we restrict the function $y(x)$ such that

$$p(x) - q(x)y(x) > 0. \tag{8}$$

Thus the cooling effect of the fin is ensured.

The solution of the problem can be made to depend on the following integral

$$E = \int_0^b \left[y\left(\frac{du}{dx}\right)^2 + (p - qy)u^2 \right] dx. \tag{9}$$

Integration by parts and use of (1), (5) and (6) gives

$$\int_0^b y\left(\frac{du}{dx}\right)^2 dx = -\int_0^b (u - 1)\,\frac{d}{dx}\left(y\,\frac{du}{dx}\right) dx = -\int_0^b (u - 1)(p - qy)u\,dx.$$

Substituting this in (9) shows that

$$E = \int_0^b (p - qy)u\,dx. \tag{10}$$

In other words, if u satisfies (1), (5) and (6), then $E = H$.

Equation (10) suggests that we minimize the integral E, treating u and y as independent variables, not supposing that u and y necessarily satisfy Eqs. (1), (5) and (6). Note that under the conditions (3) and (8), the integrand in (9) is never negative and hence E has a finite lower bound.

We proceed formally, considering a variation of $E(u, y)$ due to a variation of u and y from the assumed minimizing values. First only u is varied. The Euler equation resulting for any admissible variation of u is the governing equation (1). Now consider a variation of E resulting from a variation in y. Thus

$$\delta E = \int_0^b \delta y\left[\left(\frac{du}{dx}\right)^2 - qu^2\right] dx = 0.$$

Because of condition (4), we also have

$$\delta K = \int_0^b \delta y\,dx = 0.$$

It follows that the integrands of δE and δK must be proportional. Thus

$$(du/dx)^2 - qu^2 = \lambda. \tag{11}$$

Integrating Eq. (1) and use of (6) gives

$$y \frac{du}{dx} = -\int_x^b (p - qy)u \, dx. \tag{12}$$

In view of relations (3), (8) and (12), Eq. (11) can be written as

$$du/dx = -(qu^2 + \lambda)^{1/2}, \tag{13}$$

where λ is a constant (Lagrange multiplier). The minimum of E is obtained from the functions u and y which simultaneously satisfy Eqs. (1), (5), (6) and (13).

Because the non-linearity of Eq. (13) prevents us from finding explicit solutions, a general procedure for solving the optimum fin problem will be given before specific examples are discussed.

CASE (a). *Optimum fin.* After solving Eqs. (1), (5), (6) and (13), we will obtain a one parameter family of solutions $u(x, \lambda)$ and $y(x, b, \lambda)$ which, if substituted in Eqs. (4) and (7), give $K = K(b, \lambda)$ and $H = H(b, \lambda)$. Hence we have H as a function of b and λ which in turn are connected by K. Maximizing H while holding K constant will give the optimum fin length. The optimizing parameter λ can then be evaluated from $K(b, \lambda)$.

An alternate method which gives rather simple conditions for the optimum solutions is obtained as follow. In deriving the above variational result, the fin length b was held fixed. However, in the problem stated, b should be allowed to vary also. For this purpose we consider the function F, which is equivalent to the integral E and the subsidiary condition (4), defined by

$$F(u, y, b) = E - \lambda K = \int_0^b \left[y\left(\frac{du}{dx}\right)^2 + pu^2 - qyu^2 - \lambda y \right] dx.$$

The variation in F resulting from variations in u, y and b is

$$\delta F = 2 \int_0^b \left[y \frac{du}{dx} \left(\frac{d\,\delta u}{dx}\right) + (p - qy)u\,\delta u \right] dx + \int_0^b \delta y\left[\left(\frac{du}{dx}\right)^2 - qu^2 - \lambda \right] dx + \left[y\left(\frac{du}{dx}\right)^2 + (p - qy)u^2 - \lambda y \right] \delta x \Bigg|_0^b .$$

Integrating by parts gives

$$\int_0^b y \frac{du}{dx} \left(\frac{d\,\delta u}{dx}\right) dx = y \frac{du}{dx}\,\delta u \Bigg|_0^b - \int_0^b \delta u \frac{d}{dx}\left(y \frac{du}{dx}\right) dx.$$

The integrated part vanishes on account of (6) and $\delta u = 0$ at $x = 0$. Substituting this into the expression for δF gives

$$\delta F = -2 \int_0^b \delta u\left[\frac{d}{dx}\left(y \frac{du}{dx}\right) - (p - qy)u \right] dx + \int_0^b \delta y\left[\left(\frac{du}{dx}\right)^2 - qu^2 - \lambda \right] dx + \left[y\left(\frac{du}{dx}\right)^2 + (p - qy)u^2 - \lambda y \right] \delta x \Bigg|_0^b .$$

Of course $\delta F = 0$ for any admissible variations δu, δy and δx. The two integrands on the right give the results already obtained previously. The integrated part vanishes at the lower limit because $\delta x = 0$ at $x = 0$. At the upper limit, $\delta x \neq 0$. Use of (13) gives

$$y(du/dx)^2 + (p - qy)u^2 - \lambda y = pu^2 = 0 \quad \text{at} \quad x = b.$$

Therefore,

$$u = 0 \quad \text{at} \quad x = b. \tag{14}$$

Use of (6), (13) and (14) gives

$$y = 0 \quad \text{at} \quad x = b. \tag{15}$$

Because of (2), (1) can be written as

$$y \frac{d^2u}{dx^2} + \frac{dy}{dx}\frac{du}{dx} = (p - qy)u.$$

From (14) and (15), it is seen that

$$dy/dx = 0 \quad \text{at} \quad x = b. \tag{16}$$

Thus Eqs. (1), (5), (13), (14) [or (16)] and (15) in conjunction with (4) furnish the complete solution for the optimum fin problem.

CASE (b). *Fixed length fin.* In practical design, sometimes we wish to restrict the length of the fin. If the maximum length permitted is B, the additional restriction is

$$b \leq B. \tag{17}$$

The solution from case (a) is valid provided the optimizing value of b satisfies (17). Otherwise the solution required is simply for the case of fixed $b = B$. The procedure for affecting a solution is already stated in the first sentence of case (a) except letting $b = B$. The parameter λ in $u(x, \lambda)$, $y(x, B, \lambda)$ and $H(B, \lambda)$ is determined from $K(B, \lambda)$.

3. Heat generation independent of position. If q is assumed to be constant, Eq. (13) can readily be solved. For convenience, we define the following notations

$$\begin{aligned}
\alpha &= q^{1/2}, \\
A &= 1 + (1 + \lambda/\alpha^2)^{1/2}, \\
\Phi(\eta, -) &= Ae^{-\alpha\eta} - \lambda e^{\alpha\eta}/(\alpha^2 A), \\
\Phi(\eta, +) &= Ae^{-\alpha\eta} + \lambda e^{\alpha\eta}/(\alpha^2 A), \\
\Psi(\eta, -) &= \Phi(\eta, -)\Phi(\eta, +), \\
\Psi(\eta, +) &= A^2 e^{-2\alpha\eta} + \lambda^2 e^{2\alpha\eta}/(\alpha^4 A^2).
\end{aligned}$$

(i) Fixed length fin. From (13) and (5)

$$u = \tfrac{1}{2}\Phi(x, -) \tag{18}$$

use of (18) in (1) with (6) results in

$$y = \alpha^{-1}[\Phi(x, +)]^{-2} \int_x^B p(x)\Psi(x, -)\, dx. \tag{19}$$

The parameter λ is determined from (4) which becomes

$$K = \tfrac{1}{2}\alpha^{-1}(\alpha^2 + \lambda)^{-1/2} \int_0^B p(x)\Phi(x, -) \sinh \alpha x\, dx. \tag{20}$$

Then from (7)

$$H = E = \tfrac{1}{4}\alpha(\alpha^2 + \lambda)^{-1/2} \int_0^B p(x)\Psi(x, -)\, dx. \tag{21}$$

As $\alpha \to 0$, Eqs. (18), (19), (20) and (21) become

$$u = 1 - \lambda^{1/2}x, \tag{18a}$$

$$y = \int_x^B (\lambda^{-1/2} - x)p(x)\, dx, \tag{19a}$$

$$K = \int_0^B (\lambda^{-1/2} - x)p(x)x\, dx, \tag{20a}$$

$$H = \int_0^B (1 - \lambda^{1/2}x)p(x)\, dx, \tag{21a}$$

which except for a slight change of notation are the same results obtained from Schmidt's criterion [2].

(ii) Optimum fin. Equation (18) is still valid. Substituting (14) in (18) gives

$$(\lambda/\alpha^2)_{\text{optimum}} = (\sinh \alpha b)^{-2}.$$

Substituting this relation in Eqs. (18), (19), (20) and (21), and changing B into b give the corresponding relations for optimum fins

$$u = \sinh \alpha(b - x)/\sinh \alpha b, \tag{22}$$

$$y = \tfrac{1}{2}\alpha^{-1}\, [\cosh \alpha(b - x)]^{-2} \int_x^b p(x) \sinh 2\alpha(b - x)\, dx, \tag{23}$$

$$K = \alpha^{-2}\, (\cosh \alpha b)^{-1} \int_0^b p(x) \sinh \alpha(b - x) \sinh \alpha x\, dx, \tag{24}$$

$$H = (\sinh 2\alpha b)^{-1} \int_0^b p(x) \sinh 2\alpha(b - x)\, dx. \tag{25}$$

It can easily be shown that as $\alpha \to 0$

$$u = 1 - x/b, \tag{22a}$$

$$y = \int_x^b (b - x)p(x)\, dx, \tag{23a}$$

$$K = \int_0^b (b - x)p(x)x\, dx, \tag{24a}$$

$$H = \int_0^b (1 - x/b)p(x)\, dx, \tag{25a}$$

which are the results from Schmidt's criterion [2].

4. Rectangular fin. The variational result just obtained is now applied to specific examples of cooling fins. First we consider a fin in the shape of a rectangular plate whose sides are of lengths a and b. Let x denote the distance to one of the sides of length a. The thickness of the plate is taken to be given by a function $y(x)$. Heat is supplied at constant temperature to the side at $x = 0$. By a suitable choice of units, the temperature at this edge may be taken as unity, and the ambient temperature may be taken as zero. Let k be the coefficient of thermal conductivity of the material of the plate, h be the combined radiation and convection coefficient of the surface of the plate, and g be the constant heat generated per unit volume of the plate per unit temperature change. It is then customary to assume that the temperature $u(x)$ satisfies

$$\frac{d}{dx}\left[ky(x)\frac{du}{dx}\right] = 2hu - gyu. \tag{26}$$

Clearly, Eq. (26) can be written in the form (1) with $p = 2h/k = c$, where c denotes another constant and with $q = g/k$. Moreover, u satisfies both boundary conditions (5) and (6).

A question in the design of such a fin is the choice of $y(x)$ and b to maximize the heat conducted out of the fin base at $x = 0$ for a given weight fin. On account of (12), this heat is

$$-kay\frac{du}{dx}\bigg|_{x=0} = ak\int_0^b (p - qy)u\,dx = akH. \tag{27}$$

The volume of the fin is $a\int_0^b y\,dx = aK$. Hence the solution is for the case when p is a constant.

Substituting c for p in Eqs. (23), (24) and (25) gives the solution for the optimum fin

$$y = \tfrac{1}{2}c\alpha^{-2}\tanh^2\alpha(b - x), \tag{28}$$

$$H = \tfrac{1}{2}c\alpha^{-1}\tanh\alpha b = \tfrac{1}{2}cb - \alpha^2K, \tag{29}$$

where b is determined from the transcendental equation

$$\alpha b - \tanh\alpha b = 2\alpha^3K/c. \tag{30}$$

If the maximum length permitted is B and if B is smaller than the b determined from (30), the solution is then obtained from Eqs. (19), (20) and (21)

$$y = \tfrac{1}{2}c\alpha^{-2}[\Psi(x, +) - \Psi(B, +)]/[\Phi(x, +)]^2, \tag{31}$$

$$H = (1/8)c(\alpha^2 + \lambda)^{-1/2}[2(2 + \lambda/\alpha^2) - \Psi(B, +)], \tag{32}$$

where λ is determined from

$$(A/\lambda)\Phi(B, +)[e^{-\alpha B} - \tfrac{1}{2}\alpha(\alpha^2 + \lambda)^{-1/2}\Phi(B, +)] = (4\alpha K/c) - 2B/\alpha. \tag{33}$$

5. The circular fin. We now consider a fin in the shape of a circular plate. Let $t(r)$ be the plate thickness where r denotes the radial distance. Then the basic differential equation is

$$\frac{d}{dr}\left[krt(r)\frac{du}{dr}\right] = 2hru - grtu. \tag{34}$$

Heat is supplied to this fin at an inner radius, say $r = a$. It is then convenient to introduce the variable $x = r - a$, which gives the distance to the inner radius. The differential equation becomes

$$\frac{d}{dx}\left[(x + a)t\frac{du}{dx}\right] = c(x + a)u - q(x + a)tu. \tag{35}$$

The equation is in the form (1) with $y = (x + a)t$ and $p = c(x + a)$. The boundary conditions (5) and (6) apply as before if we take the outer radius to be $a + b$.

The cooling rate of the fin is given by $2\pi k \int_0^b (p - qy)u\, dx = 2\pi kH$. The volume of the fin is $2\pi \int_0^b y\, dx = 2\pi K$. Again, the solution is given by the relations in Sec. 3.

Substituting $p = c(x + a)$ in Eqs. (23), (24) and (25) results in the solution for an optimum fin

$$y = \tfrac{1}{4}c\alpha^{-2}[2(x + a) \tanh^2 \alpha(b - x) + \alpha^{-1} \tanh \alpha(b - x) - (b - x) \operatorname{sech}^2 \alpha(b - x)], \quad (36)$$

$$H = \tfrac{1}{4}c\alpha^{-2}(1 + 2\alpha a \tanh \alpha b - \alpha b \operatorname{sech} \alpha bc \operatorname{sech} \alpha b), \quad (37)$$

where b is calculated from

$$(b + 2a)(\alpha b - \tanh \alpha b) = 4\alpha^3 K/c. \quad (38)$$

If there is a maximum permitted radius, say $a + B$, then Eqs. (19), (20) and (21) can be applied as before.

References

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