SCATTERING OF A COMPRESSIONAL WAVE BY A PROLATE SPHEROID*

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I. Introduction. In order to estimate the size of damaged regions produced in single crystal silicon samples by collimated neutron irradiation, Truell [1] utilized a multiple scattering of elastic waves calculation made by Waterman [2] to interpret ultrasonic attenuation and velocity measurements. The multiple scattering calculation assumes a distribution of spherical scatterers and requires detailed knowledge of the scattering behavior of a single scatterer embedded in an infinite, homogeneous, isotropic medium. The problem of scattering by a single spherical scatterer of a plane compressional wave which propagates in such a medium has been studied recently by Ying and Truell [3]. The analogous problem of scattering of a plane transverse wave has been solved by Einspruch, *et al.* [4].

Examination of the single scatterer calculations reveals that the process in which a portion of the energy in an incident compressional wave becomes transferred into energy transported by a scattered shear wave (i.e. mode conversion) is a process of higher order; the transfer of energy from an incident compressional wave to a scattered compressional wave dominates mode conversion. The equivalent statement for scattering of a shear wave holds as well. Since mode conversion can be regarded as a perturbation to the scattering process, an approximate description of scattering of a compressional wave can be made by replacing the elastic medium in which the scatterer occurs by a compressible fluid (i.e. acoustic) medium.

Since radiation damage in solids [5] frequently occurs as highly localized regions which are non-spherical, a prolate spheroid may prove to be a more accurate representation of the shape of a damaged region than a sphere. This is the motivation for the solution of the problem presented here.

In the work of Ying and Truell and of Einspruch, *et al.* the normalized scattering cross section of a spherical obstacle which was a cavity or a mismatched elastic medium was found to depend on $(Ka)^4$ ($K = 2\pi/\lambda$; λ = wavelength of incident wave; a = radius of scatterer) in the Rayleigh limit ($Ka \ll 1$). In the case of scattering by a rigid fixed sphere, the normalized scattering cross section was found to depend on the sum of a frequency and size independent term and a term which varies as $(Ka)^4$. Since one would expect the scattering cross section to vanish in the limit of large wavelengths, the rigid obstacle is probably a poor choice of model for the description of the actual physical situation.

The problem of scattering of a plane compressional wave by a rigid prolate spheroid has been solved by Spence and Granger [6]. In the present paper, solutions to the problems of scattering by a soft acoustic scatterer and by a small acoustic scatterer of arbitrary properties are presented.

II. Theory. The definition of the prolate spheroidal co-ordinate system is

$$x = f \cos\phi[(\xi^2 - 1)(1 - \eta^2)]^{1/2},$$

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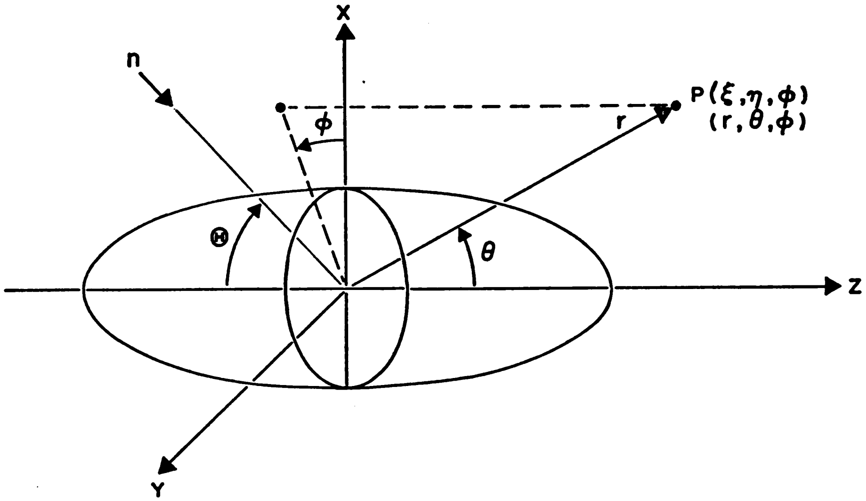

FIG. 1. Prolate spheroidal co-ordinate system.

$$y = f \sin \phi[(\xi^2 - 1)(1 - \eta^2)]^{1/2},$$

$$z = f\xi\eta,$$

where f is the semifocal distance, $0 \leq \phi \leq 2\pi$, $1 \leq \xi \leq \infty$, $-1 \leq \eta \leq 1$ (see Fig. 1). The mathematical problem is to find solutions of the scalar Helmholtz equation

$$\nabla^2\psi + K^2\psi = 0$$

within and external to the scatterer which satisfy the boundary conditions at the interface between the scatterer and the surrounding fluid medium; ψ is the velocity potential function; simple harmonic time dependence is assumed throughout this discussion.

The incident wave is a plane wave of unit amplitude with wave normal in the xz plane at an angle Θ with the positive z axis. The expansion [7] for the velocity potential, ψ_i, for the incident plane wave is

$$\psi_i = 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} S_{ml}(\eta) S_{ml} (\cos \Theta) R_{ml}^{(1)}(\xi) \cos m\phi, \tag{1}$$

where

$R_{ml}^{(1)}$ = radial function of the first kind

S_{ml} = angular function of the first kind

N_{ml} = norm of S_{ml}

$$\epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0. \end{cases}$$

A thorough discussion of the properties of the prolate spheroidal co-ordinate system and the associated wave functions can be found in the text by Morse and Feshbach [7].

The radial and angular functions depend on the parameter Kf; wherever it is convenient, this parameter is supressed. The potential function for the wave scattered by the obstacle must satisfy the Helmholtz equation, the boundary conditions, and the radiation condition

$$\psi_s \underset{r\to\infty}{\longrightarrow} f(\theta, \phi) \frac{e^{iKr}}{r};$$

$f(\theta, \phi)$ is the scattering amplitude.

The potential function for the scattered wave is constructed as follows

$$\psi_s = 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} A_{ml} S_{ml}(\eta) R_{ml}^{(3)}(\xi) \cos m\phi, \tag{2}$$

where

A_{ml} = expansion coefficients

$R_{ml}^{(3)}$ = radial functions of the third kind.

The A_{ml} are determined by the boundary conditions at the surface of the scatterer.

III. Scattering by a soft obstacle. Since a soft obstacle will not support a pressure, the boundary condition which must hold at the interface between such an obstacle and the fluid in which it occurs is

$$\psi_i + \psi_s = 0, \qquad \xi = \xi_0$$

since the pressure produced by each wave is $\rho\, \partial\psi/\partial t$, where ρ is the density of the medium.

This boundary condition yields

$$A_{ml} = -S_{ml} (\cos \Theta) \frac{R_{ml}^{(1)}(\xi_0)}{R_{ml}^{(3)}(\xi_0)}.$$

The phase shifts are defined as

$$\delta_{ml} = \tan^{-1} \frac{R_{ml}^{(1)}(\xi_0)}{R_{ml}^{(2)}(\xi_0)},$$

where

$R_{ml}^{(2)}$ = radial functions of the second kind.

The expansion coefficients are thus given by

$$A_{ml} = -i \sin \delta_{ml} e^{-i\delta_{ml}} S_{ml} (\cos \Theta).$$

Since

$$R_{ml}^{(3)}(\xi) \underset{r\to\infty}{\longrightarrow} -i(-i)^{l+m} \frac{e^{iKr}}{Kr}$$

and at large r

$$\eta \longrightarrow \cos \theta$$

$$f(\theta, \phi) = -\frac{2}{K} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-i)^m \frac{\epsilon_m}{N_{ml}} \sin \delta_{ml} \exp(-i\delta_{ml}) \cos m\phi S_{ml} (\cos \Theta) S_{ml} (\cos \theta).$$

The total scattering cross section, σ, is defined as

$$\sigma = \int_0^{2\pi} \int_0^{\pi} |f(\theta, \phi)|^2 \sin \theta \, d\theta \, d\phi.$$

The integration yields

$$\sigma = \frac{8\pi}{K^2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_m^2}{N_{ml}} \sin^2 \delta_{ml} [S_{ml} (\cos \Theta)]^2;$$

this expression is exact for all wavelengths.

In the short wavelength limit

$$R_{ml}^{(1)}(Kf, \xi) \underset{Kf\to\infty}{\longrightarrow} \frac{1}{Kf\xi} \sin \left(Kf\xi - \frac{l+m}{2}\pi\right)$$

and

$$R_{ml}^{(2)}(Kf, \xi) \underset{Kf\to\infty}{\longrightarrow} -\frac{1}{Kf\xi} \cos \left(Kf\xi - \frac{l+m}{2}\pi\right),$$

consequently

$$\delta_{ml} \underset{Kf\to\infty}{\longrightarrow} -\left(Kf\xi_0 - \frac{l+m}{2}\pi\right).$$

The total scattering cross section is thus given by

$$\sigma = \frac{8\pi}{K^2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_m^2}{N_{ml}} [S_{ml} (\cos \Theta)]^2 \sin^2 \left(Kf\xi_0 - \frac{l+m}{2}\pi\right).$$

In the long wavelength limit

$$S_{ml} (\cos \theta) \underset{Kf\to 0}{\longrightarrow} P_l^m (\cos \theta)$$

and

$$N_{ml} \underset{Kf\to 0}{\longrightarrow} \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!};$$

where the $P_l^m (\cos \theta)$ are the associated Legendre polynomials.
The total scattering cross section is thus given by

$$\sigma = \frac{8\pi}{K^2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \epsilon_m^2 \sin^2 \delta_{ml} [P_l^m (\cos \theta)]^2.$$

IV. Scattering by an obstacle of arbitrary acoustic properties. In the problem considered in Sec. III, the common technique of equating the coefficients of the angular functions which appear in the equation derived from the boundary conditions was used to evaluate the expansion coefficients. In the case of the soft scatterer, no wave is excited within the scatterer; hence the only waves which propagate are external to the scatterer. In the prolate spheroidal coordinate system, the wave number appears as a parameter in the angular functions as well as in the radial functions. Consequently, the technique utilized previously is not applicable if a wave propagates within the scatterer except in the Rayleigh limit, as shall be demonstrated.

The solution for $\xi > \xi_0$ is the sum of the incident wave, Eq. (1), and a scattered wave analogous to Eq. (2).

$$\psi_i = 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} S_{ml}(K_2, \cos \Theta) S_{ml}(K_2, \eta) R_{ml}^{(1)}(K_2, \xi) \cos m\phi$$

$$\psi_s = 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} A_{ml} S_{ml}(K_2, \eta) R_{ml}^{(3)}(K_2, \xi) \cos m\phi,$$

in these and the following expressions the dependence on the wave number is indicated. Subscript 2 indicates a property of the outer medium; subscript 1 indicates a property of the scatterer.

For $\xi < \xi_0$, the solution is given by

$$\psi_e = 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} B_{ml} S_{ml}(K_1, \eta) R_{ml}^{(1)}(K_1, \xi) \cos m\phi.$$

In the Rayleigh limit

$$\psi_i \underset{K_2 f \to 0}{\longrightarrow} 2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i)^l \frac{\epsilon_m}{N_{ml}} P_l^m (\cos \Theta) P_l^m(\eta) R_{ml}^{(1)}(K_1, \xi) \cos m\phi \tag{3}$$

and similarly for ψ_s and ψ_e.

The boundary conditions which must be satisfied at $\xi = \xi_0$ are continuity of pressure and normal component of fluid particle velocity

$$\left.\begin{aligned} \rho_2 \psi_i + \rho_2 \psi_s &= \rho_1 \psi_e \\ \frac{\partial \psi_i}{\partial \xi} + \frac{\partial \psi_s}{\partial \xi} &= \frac{\partial \psi_e}{\partial \xi} \end{aligned}\right\} \xi = \xi_0 . \tag{4}$$

Substitution of Eqs. (3) and Eqs. (4) and performance of the indicated operations yield

$$A_{ml} = P_l^m (\cos \Theta)[\rho_2 R_{ml}^{(1)}(K_2, \xi_0) R_{ml}^{(1)\prime}(K_1, \xi_0) - \rho_1 R_{ml}^{(1)\prime}(K_2, \xi_0) R_{ml}^{(1)}(K_1, \xi_0)] \Delta^{-1},$$

$$B_{ml} = P_l^m (\cos \Theta)[\rho_2 R_{ml}^{(1)}(K_2, \xi_0) R_{ml}^{(3)\prime}(K_2, \xi_0) - \rho_2 R_{ml}^{(3)}(K_2, \xi_0) R_{ml}^{(1)\prime}(K_2, \xi_0)] \Delta^{-1},$$

where

$$\Delta = \rho_1 R_{ml}^{(1)}(K_1, \xi_0) R_{ml}^{(3)\prime}(K_2, \xi_0) - \rho_2 R_{ml}^{(3)}(K_2, \xi_0) R_{ml}^{(1)\prime}(K_1, \xi_0).$$

The prime indicates differentiation with respect to ξ.

In terms of the expansion coefficients for the scattered wave, the scattering amplitude is given by

$$f(\theta, \phi) = -\frac{2i}{K} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-i)^m \frac{\epsilon_m}{N_{ml}} A_{ml} S_{ml}(K_2, \eta) \cos m\phi.$$

As in Sec. III, the total scattering cross section is obtained by integrating the scattering amplitude over a large sphere concentric with the spheroidal scatterer

$$\sigma = \frac{8\pi}{K^2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_m^2}{N_{ml}} \mid A_{ml} \mid^2.$$

V. Summary. In this paper the authors have solved the problem of scattering of a plane acoustic wave by a soft prolate spheroidal obstacle embedded in a fluid medium. An exact solution is given as well as approximate solutions which are valid in the long and short wavelength limits. A solution is also presented for the problem of scattering of a compressional wave by a small prolate spheroidal obstacle of arbitrary acoustic properties. In each case, an expression for the total scattering cross section is derived.

References

1. R. Truell, *Nature of defects arising from fast neutron irradiation of silicon single crystals,* Phys. Rev. **116,** 890–892 (1959)
2. P. C. Waterman, *Multiple scattering of waves,* Ph.D. thesis, Brown University, 1958 (unpublished)
3. C. F. Ying and R. Truell, *Scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid,* J. Appl. Phys. **27,** 1086–1097 (1956)
4. N. G. Einspruch, E. J. Witterholt and R. Truell, *Scattering of a plane transverse wave by a spherical obstacle in an elastic medium,* J. Appl. Phys. **31,** 806–818 (1960)
5. See, for example: G. J. Dienes and G. H. Vineyard, *Radiation effects in solids,* Interscience Publishers, Inc., New York, 1957
6. R. D. Spence and S. Granger, *The scattering of sound from a prolate spheroid,* J. Acoust. Soc. Amer. **23,** 701–706 (1951)
7. P. M. Morse and H. Feshbach, *Methods of theoretical physics,* McGraw-Hill Book Co., Inc., New York, 1953; contains explanations of many of the mathematical steps indicated in this paper.