ON "TRANSCRITICAL" AND "HYPERCRITICAL" FLOWS IN MAGNETOGASDYNAMICS*

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Introduction. A category of magnetogasdynamic flows that has attracted considerable interest involves the steady motion of a perfectly conducting, inviscid, compressible fluid with magnetic field everywhere aligned with the flow direction. This type of flow must result if the magnetic field and flow direction are aligned anywhere in plane and axially symmetric configurations. It is clear that such flows are isentropic, or in the case of flow from a uniform state, homentropic.

The system of equations governing motion in this category may be either elliptic or hyperbolic, depending upon the local values of the Alfvén number, A, and the Mach number, M. The Alfvén number is defined here as the ratio of the flow speed to the propagation speed of Alfvén waves, while the Mach number is the usual ratio of flow speed to the speed of sound in the absence of magnetogasdynamic effects. These elliptic and hyperbolic regions were first exhibited by Taniuti [1] in a diagram similar to Fig. 1, and later by Kogan [2] and Resler (see Sears [3]). There are two regions where the equations are hyperbolic, one of which is subsonic, sub-Alfvénic ($A < 1$), the other being supersonic, super-Alfvénic ($A > 1$). In these regions the proper family of characteristics must be chosen. Sears [4] has pointed out that for the subsonic case waves which propagate upstream are the correct choice, while in the supersonic region it is the usual downstream-propagating waves.

Along PR, where $A^2 + M^2 = 1$, the propagation speed of small magnetosonic disturbances vanishes. The elliptic region OPR where the velocity is less than the propagation speed will be referred to as *subcritical.* The subsonic, hyperbolic region PRQ where the velocity is greater than this speed will be called *supercritical.* Near PR the flow will be termed *hypercritical.* Note that both transonic and trans-Alfvénic flows are, under this nomenclature, *transcritical.*

McCune and Resler [5] have derived the linear theories for this type of flow. Just as the results of ordinary linearized gasdynamics become invalid in the transonic regimes, McCune and Resler's results break down in the transonic, trans-Alfvénic and hypercritical regimes. The object here is to study the flow in these regimes where the motion is fundamentally non-linear and the non-linear terms of the equations must be retained. Of particular interest is the hypercritical regime where elliptic flow joins with hyperbolic flow having forward-facing characteristics.

It is found that for the flow under consideration the hodograph transformation can easily be effected, the result being two second-order, linear, partial differential equations analogous to Chaplygin's equation and the potential equation for ordinary compressible flow. Two elementary solutions of these equations, corresponding to source and vortex motion, are discussed. One displays smooth transition through the hypercritical and trans-Alfvénic regimes. Separation of variables is possible, and the resulting

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ordinary differential equations are discussed briefly. In the special case where the ratio of specific heats, γ, is 2, the equations are of the Fuchsian type with five singularities.

Equations of motion. The equations governing the steady flow of an ideal conductor in the presence of a steady magnetic field are

$$\operatorname{div} \rho\mathbf{q} = 0, \tag{1}$$

$$\operatorname{grad} q^2/2 - \mathbf{q}\times \operatorname{curl} \mathbf{q} + (\mu/4\pi\rho)\mathbf{H}\times \operatorname{curl} \mathbf{H} + (\operatorname{grad} p)/\rho = 0, \tag{2}$$

$$\operatorname{div} \mathbf{H} = 0, \tag{3}$$

$$\operatorname{curl} (\mathbf{q} \times \mathbf{H}) = 0, \tag{4}$$

$$\mathbf{q}\cdot\operatorname{grad} S = 0, \tag{5}$$

where p, ρ, S, μ are the pressure, density, entropy and magnetic permeability, and $\mathbf{q}$ and $\mathbf{H}$ the velocity and magnetic field vectors. In plane flow, Eq. (4) requires that $\mathbf{q} \times \mathbf{H}$ be constant; thus if $\mathbf{q}$ and $\mathbf{H}$ are parallel anywhere (e.g., at infinity), they are parallel everywhere. This can be expressed as

$$\rho\mathbf{q} = f\mathbf{H}, \tag{6}$$

where f is some scalar function. Equations (1) and (3) then give $f = f(\psi)$, where ψ is the stream function. Substituting Eq. (6) into (2), forming the scalar product with $\mathbf{q}$, and noting Eq. (5) gives Bernoulli's equation

$$q^2/2 + \int dp/\rho = \text{constant} \tag{7}$$

along a streamline. Assuming uniform conditions at infinity, f becomes constant, the flow is homentropic, and Eq. (7) holds through the flow. Combining the gradient of Eq. (7) with Eq. (2) and (6) then yields

$$\operatorname{curl} (1 - A^{-2})\mathbf{q} = 0, \tag{8}$$

where A denotes the Alfvén number, $(4\pi\rho/\mu)^{\frac{1}{2}}q/H$.

The characteristics for this system, first given by Taniuti [1], can be expressed in the form

$$a^2[A^2 + M^2 - 1][(dx)^2 + (dy)^2] = A^2[u\, dy - v\, dx]^2 \tag{9}$$

in the physical plane, and

$$(M^2 - 1)[A^2 + M^2 - 1][u\, dv + v\, du]^2 = (A^2 - 1)[u\, dv - v\, du]^2 \tag{10}$$

in the hodograph plane, where a^2 denotes $dp/d\rho$ and M, q/a. From Eq. (10) the diagram in Fig. 1 can easily be deduced. Equation (9) shows that as $A^2 + M^2 \to 1$, the characteristics become parallel to the streamlines, and as $A \to 1$ or $M \to 1$, they become perpendicular to the streamlines.

Hodograph transformation. Equation (8) suggests the introduction of a potential function, φ, such that

$$\operatorname{grad} \varphi = (1 - A^{-2})\mathbf{q}.$$

With this equation, the usual definition of the stream function, Bernoulli's Equation (7), and the continuity Equation (1), the transformation to the hodograph plane can be effected in a manner identical to that for the analogous equations in ordinary compressible flow (see e.g. von Mises [6]). The first-order, linear, differential equations

$$\frac{\partial\varphi}{\partial\theta} = \frac{(A^2 - 1)^2}{A^2(A^2 + M^2 - 1)} \frac{q}{\rho} \frac{\partial\psi}{\partial q}, \tag{11}$$

$$\frac{\partial\varphi}{\partial q} = \frac{(M^2 - 1)(A^2 - 1)}{A^2 \rho q} \frac{\partial\psi}{\partial\theta}, \tag{12}$$

are the result, subject to the condition that the Jacobian of the mapping does not vanish. In terms of ψ alone this requires that

$$I = \frac{\partial(x, y)}{\partial(q, \theta)} = \frac{1}{\rho^2 q^3}\left[(M^2 - 1)\left(\frac{\partial\psi}{\partial\theta}\right)^2 - \frac{(A^2 - 1)q^2}{A^2 + M^2 - 1}\left(\frac{\partial\psi}{\partial q}\right)^2\right] \neq 0. \tag{13}$$

Limit lines, i.e. curves on which $I = 0$, can occur only in the two hyperbolic regions. In contradistinction to ordinary compressible flow, the Jacobian

$$D = \frac{\partial(\varphi, \psi)}{\partial(q, \theta)} = \rho q^2 \frac{(A^2 - 1)}{A^2} I$$

can be zero even though I is not zero. This is a result of the definition of φ for this flow.

Elimination of the potential function from Eqs. (11) and (12) gives, for a perfect gas,

$$\begin{aligned}(M^2 - 1) \frac{[A^2 + M^2 - 1]^2}{q^2} \frac{\partial^2\psi}{\partial\theta^2} - (A^2 - 1)(A^2 + M^2 - 1) \frac{\partial^2\psi}{\partial q^2} \\ - \{(1 + M^2)(A^2 - 1)^2 - M^4[\gamma(A^2 - 1) + 1 - 3A^2]\} \frac{1}{q} \frac{\partial\psi}{\partial q} = 0,\end{aligned} \tag{14}$$

which reduces to Chaplygin's equation in the limit of no magnetic field, $A \rightarrow \infty$. The equation for φ alone is

$$\begin{aligned}(1 - M^2)^2 \frac{[A^2 + M^2 - 1]}{q^2} \frac{\partial^2\varphi}{\partial\theta^2} + (1 - M^2)(A^2 - 1) \frac{\partial^2\varphi}{\partial q^2} \\ + [(1 + \gamma M^4)(A^2 - 1) + M^2(M^2 - 1)] \frac{1}{q} \frac{\partial\varphi}{\partial q} = 0.\end{aligned} \tag{15}$$

To return to the physical plane the following coordinate relations are required

$$dx = \frac{1}{q}\left[\frac{A^2}{A^2 - 1} \cos\theta \, d\varphi - \frac{\sin\theta}{\rho} d\psi\right], \tag{16}$$

$$dy = \frac{1}{q}\left[\frac{\cos\theta}{\rho} d\psi + \frac{A^2}{A^2 - 1} \sin\theta \, d\varphi\right]. \tag{17}$$

Elementary solutions. Equations (14) and (15) have the "elementary" solutions $\psi = K\theta$ and $\varphi = K\theta$, where K is a constant. The corresponding solutions in the physical plane could have been obtained directly from Eqs. (1) and (8), without recourse to the hodograph transformation.

The first of these solutions, which might be termed "magnetic-source" flow, has a two-fold mapping to the physical plane and the limit line $M = 1$. Along this line the two solutions, one subsonic and the other supersonic, join. These are depicted in Fig. 1 by the curves Ia and Ib. The arrows indicate the direction of increasing radius. Equa-

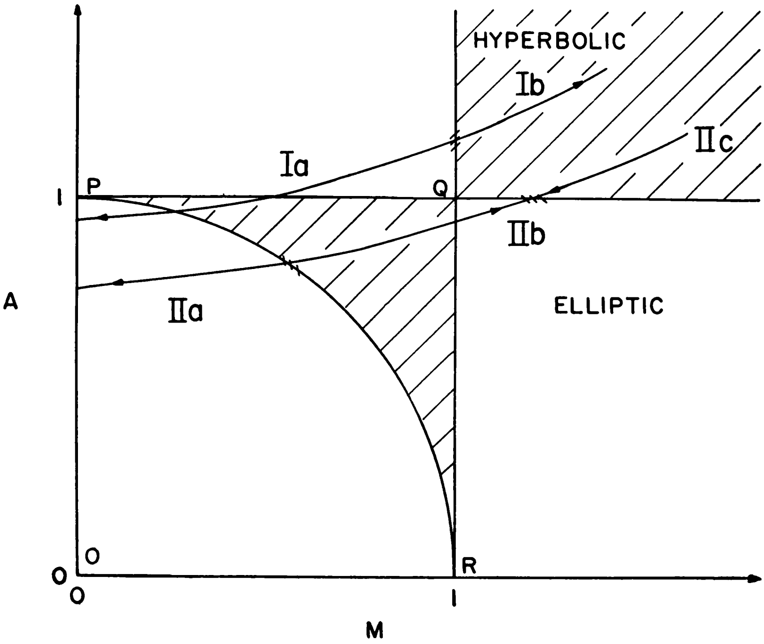

FIG. 1. Diagram of elliptic and hyperbolic regions of flow.

tions (16) and (17) give the coordinate functions x and y, which are identical to the non-magnetic results and will not be given here (see e.g. [4]). The subsonic flow, Ia, which undergoes smooth transition through the hypercritical and trans-Alfvénic regimes, is shown in Fig. 2.

The second solution, a "magnetic-vortex" flow, has the limit line $A^2 + M^2 = 1$ and the coordinate function

$$r = (x^2 + y^2)^{1/2} = \pm \frac{KA^2}{q(A^2 - 1)}.$$

In this case the mapping to the physical plane is threefold, the flows being indicated in Fig. 1 by the Curves IIa, IIb, and IIc. One set of flows joins at the limit line, the other at a branch line $1/I = 0$. The flow corresponding to IIb is shown in Fig. 3.

The special case $\gamma = 2$. Assuming a solution to Eq. (14) of the form $\psi = f(\theta)F(\tau)$ results in

$$f = c_1 e^{ik\theta} + c_2 e^{-ik\theta} \tag{18}$$

and a second-order, linear, differential equation for F, where $\tau = q^2/q^2_{\text{Max}}$ and k is the separation constant. The coefficients of this equation are, in general, complicated algebraic functions of τ. With $\gamma = (n + 1)/n$, n a positive integer, the coefficients become polynomials in τ, the degree of which depends on n. In the simplest case n is 1 and the equation for F becomes

$$\frac{d}{d\tau}\left[\frac{\tau(\tau - \beta)^2}{(\tau - \beta/3)(\tau - 1)} \frac{dF}{d\tau}\right] + \frac{3k^2}{4} \frac{(\tau - \beta)(1 - 3\tau)}{\tau(\tau - 1)^2} F = 0, \tag{19}$$

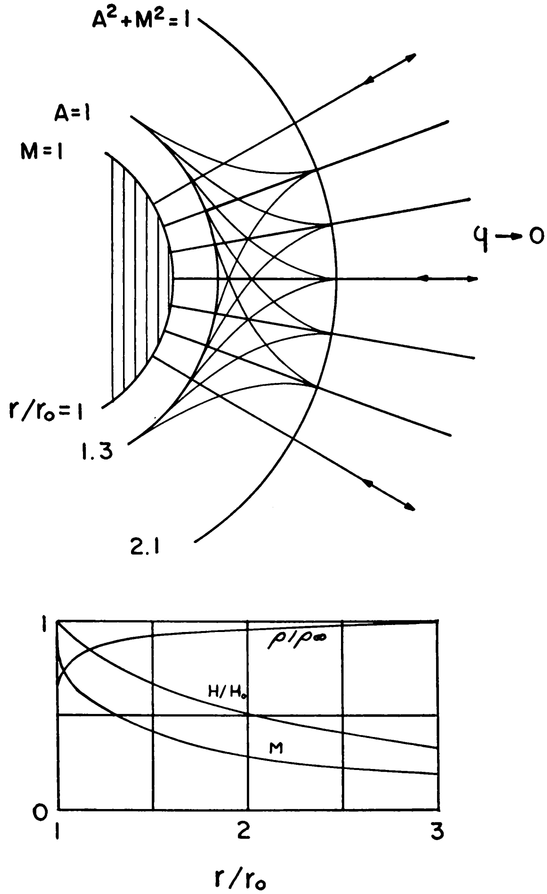

FIG. 2. Streamlines and characteristics for "magnetic-source" flow.

where $1 - \beta = A^2$ evaluated at $\tau = 0$. Here there are five singularities, all regular, and thus the equation is of the Fuchsian type. In terms of the extended Riemann P-function [7],

$$F = P\left\{\begin{matrix} 0 & \beta/3 & \beta & 1 & \infty & \\ -k/2 & 0 & 0 & 0 & -3k/2 & \tau \\ +k/2 & 2 & -1 & 2 & +3k/2 & \end{matrix}\right\}. \tag{20}$$

The singularities $\beta/3$ and β correspond to $A^2 + M^2 = 1$ and $A^2 = 1$ respectively. Results analogous to Eq. (18), (19) and (20) may also be obtained for the potential function. They are more complicated in that the exponents at two of the singularities are functions of β.

An interesting conclusion can be drawn in this case with regard to the flow behavior as $A^2 + M^2 \to 1$. The singularity at $\tau = \beta/3$ can be shown to be apparent, and therefore both solutions are analytic about this point. This insures further that I will be bounded and generally non-zero. In the hypercritical regime then, for $\gamma = 2$, the flow behavior

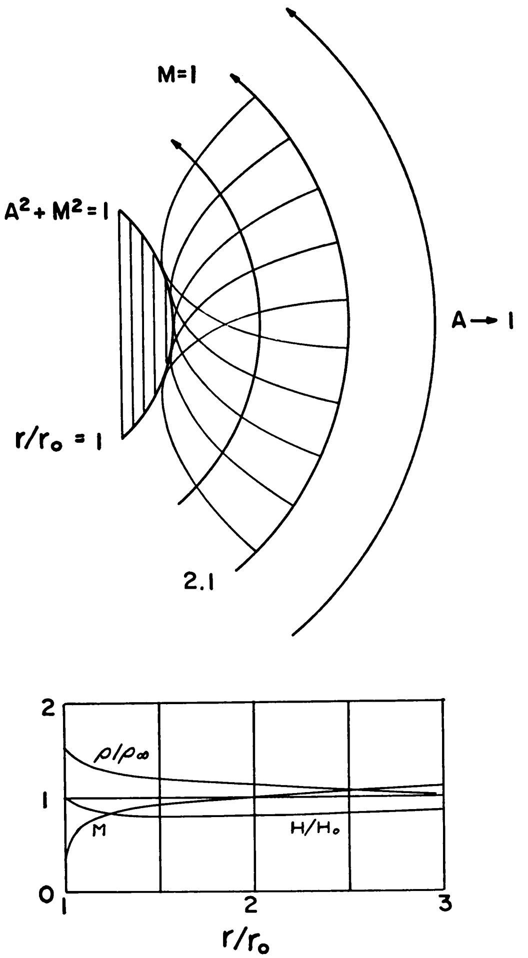

FIG. 3. Streamlines and characteristics for "magnetic-vortex" flow.

must be smooth. On physical grounds it might be conjectured that this will hold for other values of γ. In contrast, 0 and 1 are the only real values of k for which $\tau = 1$ is an apparent singularity.

It is possible to solve Eq. (19) in terms of rational functions of τ for particular values of k. The procedure is too lengthy for presentation here. For $k = 1$, the two solutions are

$$F = \frac{1}{\tau^{1/2}(\tau - \beta)}$$

and

$$F = \frac{\tau^{1/2}[2\tau^2 - (\beta + 3)\tau + 2\beta]}{\tau - \beta}.$$

Solutions of this class should provide insight into the flow behavior in the hypercritical regimes analogous to that obtained from the solutions of Ringleb [8] or Temple and Yarwood (see [9]) in ordinary gasdynamics.

In the limit $\beta \to 0$ Eq. (20) reduces to the Riemann P-equation with the solution

$$F = \tau^{[-1+(1+3k^2)^{1/2}]/2}$$

$$\cdot R\left(\frac{-1 - 3k + (1 + 3k^2)^{1/2}}{2}, \frac{-1 + 3k + (1 + 3k^2)^{1/2}}{2}; 1 + (1 + 3k^2)^{1/2}; \tau\right)$$

where R is the hypergeometric function. For $k = 1$ a solution is

$$\psi = \tau^{-3/2} \sin \theta, \qquad \varphi = \tau^{-1/2}(1 - \tau)^{-1} \cos \theta.$$

Since $\beta = 0$ the motion is entirely super-Alfvénic. The limit line and streamlines are similar to those of Ringleb flow.

Conclusion. Through this investigation the existence of smooth flows in the transonic, trans-Alfvénic, and hypercritical regimes has been determined. The flow behavior in the hypercritical regime will generally be regular for the special case $\gamma = 2$. The new and interesting phenomena associated with the flow of a highly conducting gas in the presence of a strong aligned magnetic field have been exemplified in two simple solutions.

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Note added in proof. The details of the flow $\psi = \tau^{-\frac{1}{2}}(\tau - \beta)^{-1} \sin \theta$ have been studied by the author and some results were presented by Prof. W. R. Sears in his paper *Some paradoxes of sub-Alfvénic flow of a compressible conducting fluid* at the Symposium on Electromagnetics and Fluid Dynamics of Gaseous Plasma, New York, N. Y., April 1961.

The author's attention has been called to two earlier papers in which equations equivalent to Eqs. (11) and (12) were given; viz.,

I. M. Iur'ev, *On the solution of the equations of magnetogasdynamics*, Prikl. Mat. i Mekh. **24**, 168 (1960).
K. Hida, *Hodograph method for treating the flow of a perfectly conducting fluid with aligned magnetic field*, Preprint from 5th Meeting on Mechanics and Applied Mathematics, at Matsuyama, Japan, (1960).
In the latter reference the solutions $\psi = k\theta$ and $\varphi = k\theta$ were also pointed out.