FLOWS OF DILATANT FLUIDS*

BY

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1. Introduction. In this paper we shall be concerned with the construction and behavior of simple models representing a class of fluids known in the literature [1] as "dilatant fluids".

A typical relation between shearing stress (s) and shear rate (e) for a dilatant fluid in simple shear[1] is represented by the diagram in Fig. 1, for a wide range of s. For shearing stresses s, in the neighborhood of $s = 0$, the behavior of the fluid is well approximated by that of a Newtonian fluid. However, after a certain shear rate $e = e_0$, corresponding to the shearing stress s_0, has been attained, e grows not nearly as rapidly with increasing s as in this Newtonian fluid.

In Fig. 2 we exhibit the idealisation of the s versus e relation of a dilatant fluid in simple shear on which models for more general states of stress will be based. According to Fig. 2 we have

$$2\mu e = s, \quad \text{for} \quad s \leq s_0 , \tag{1.1}$$

and

$$e = e_0 \quad \text{for} \quad s \geq s_0 . \tag{1.2}$$

The coefficient of viscosity μ in (1.1) is assumed constant for a given fluid.

There are many mathematically admissible relations between stress and velocity strain, for completely general states of stress, which reduce to (1.1) and (1.2) when the fluid is in simple shear. Because of the lack of experimental evidence, the motivation

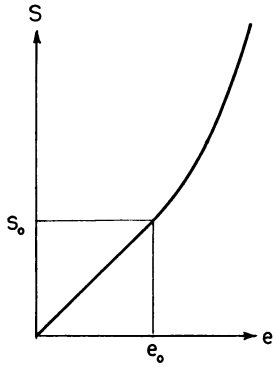

FIG. 1

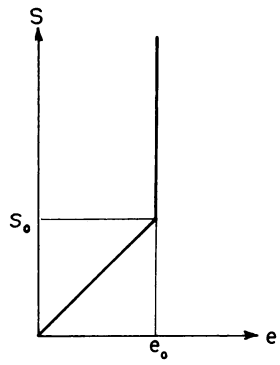

FIG. 2

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[1]An example of simple shear is the one-dimensional flow between two parallel plates which are in relative motion in a direction parallel to the plates.

for the choice of the models discussed in this paper will be their mathematical simplicity. The most that should be reasonably expected of the models is that they shall exhibit the principal qualitative features of a real dilatant fluid and hence provide a basis with which future experimental results can be compared.

It is convenient to use the Cartesian tensor notation both to define the models and to discuss their behavior. Let $x_i (i = 1, 2, 3)$ be fixed rectangular Cartesian coordinates. If $u_i(x_1, x_2, x_3, t)$ are the velocity components of the fluid particle that has the position x_i at the time t, the velocity strain tensor has components

$$e_{ij} = \tfrac{1}{2}[u_{i,j} + u_{j,i}]. \tag{1.3}$$

In (1.3) the usual indicial notation has been employed: the subscript j preceded by a comma denotes differentiation with respect to x_j.

It is convenient to introduce the scalar quantities

$$J_1 = e_{ii}, \qquad J_2 = e_{ij}e_{ji}, \quad \text{and} \quad J_3 = e_{ij}e_{jk}e_{ki}, \tag{1.4}$$

where the usual summation convention has been used. As is well known, any invariant of the velocity strain tensor can be expressed by means of J_1, J_2 and J_3. In terms of the principal components e_1, e_2, e_3 of the velocity strain tensor, these "basic invariants" are given by

$$J_1 = e_1 + e_2 + e_3, \qquad J_2 = e_1^2 + e_2^2 + e_3^2 \quad \text{and} \quad J_3 = e_1^3 + e_2^3 + e_3^3. \tag{1.5}$$

In generalising the mechanical behavior described by (1.1) and (1.2) we will assume that with every dilatant fluid there can be associated a continuous function $\varphi(e_{pq})$ of the components of the velocity strain tensor satisfying

$$\varphi(0) < 0; \tag{1.6}$$

only flows for which

$$\varphi(e_{pq}) \leq 0, \tag{1.7}$$

are admissible for the model. The function φ will be called the "dilatancy function"; it may, or may not, be analytic in the components e_{pq}. The exact form of φ for a given fluid should, of course, be determined experimentally. In what follows states for which

$$\varphi(e_{pq}) < 0, \tag{1.8}$$

are called "regular states"; condition (1.6) and the assumed continuity of φ imply that such states exist. States for which

$$\varphi(e_{pq}) = 0 \tag{1.9}$$

are called "dilatant states".

It will be assumed that, in both regular and dilatant states, the components of the velocity strain tensor at a point in the fluid are functions of the components of the total stress tensor p_{ij} at that point. Moreover in the regular state the fluid is isotropic and behaves like a Newtonian fluid. These assumptions imply that for

$$\varphi(e_{pq}) < 0, \tag{1.10}$$

we have

$$p_{ij} = -p\,\delta_{ij} + \lambda e_{kk}\,\delta_{ij} + 2\mu e_{ij}, \tag{1.11}$$

where δ_{ij} denotes the Kronecker delta. In (1.11) λ and μ are characteristic constants of the fluid; p is the thermodynamic pressure which is a specified function of the density ρ and the internal energy i per unit mass.

For a given dilatancy function φ there are many forms of the relation between p_{ij} and e_{ij} in the dilatant state which reduce to (1.1) and (1.2) in simple shear; each relation defines a model. In this paper we will discuss two possible models and certain features of their flows.

2. Constitutive equations. The relation between s and e, exhibited in Fig. 2, for a dilatant fluid in simple shear can be alternatively interpreted as the relation between axial strain (s) and axial stress (e) for an elastic, perfectly plastic solid in simple tension [2]. In this section we will develop a model of a dilatant fluid, for a general state of stress, in much the same way as the elastic, perfectly plastic model of a plastic solid is developed for a general state of strain [2].

The dilatancy function of a dilatant fluid corresponds to the yield function of an elastic, perfectly plastic solid; while the former depends on the components of the velocity strain, the latter depends on the stress components.

For our model of a dilatant fluid the stress tensor p_{ij} is supposed to depend on the velocity strain tensor e_{ij} as follows:

$$p_{ij} = -p\,\delta_{ij} + \lambda e_{kk}\,\delta_{ij} + 2\mu e_{ij} + \alpha \frac{\partial \varphi}{\partial e_{ij}}, \tag{2.1}$$

where the coefficient α in the last term vanishes when the fluid is in a regular state ($\varphi < 0$), and is positive or zero when the fluid is in a dilatant state ($\varphi = 0$). In (2.1), the dilatancy function φ is to be written symmetrically in the symmetric components e_{ij} and e_{ji}, which are formally treated as independent variables when the derivative in the last term of (2.1) is formed. This last term expresses the assumption that in a dilatant state the stress tensor can differ from the Newtonian one only by a positive multiple of the derivative $\partial\varphi/\partial e_{ij}$.

The form (2.1) of the relation between p_{ij} and e_{ij} is analogous to the 'flow rule' of elastic, perfectly plastic solids, which leads to certain uniqueness properties and extremum principles. This analogy will be further developed and exploited in what follows.

Throughout this paper we will assume that the fluid is isotropic in both regular and dilatant states. It follows from this that the dilatancy function φ is a function of J_1, J_2 and J_3 only and that stress and velocity strain at any point of the fluid have a common system of principal axes. The term $\partial\varphi/\partial e_{ij}$ in (2.1) can then be written as follows:

$$\frac{\partial \varphi}{\partial e_{ij}} = \frac{\partial \varphi}{\partial J_1}\,\delta_{ij} + 2\frac{\partial \varphi}{\partial J_2}\,e_{ij} + 3\frac{\partial \varphi}{\partial J_3}\,e_{ik}e_{kj}\,. \tag{2.2}$$

For Newtonian fluids, it is often stipulated that $-p$ is identical with the mean normal stress $p_{kk}/3$; the viscosity constants λ and μ in (1.11) must then satisfy the Stokes relation

$$3\lambda + 2\mu = 0. \tag{2.3}$$

If the same stipulation is made for the dilatant fluid with the constitutive equation (2.1), the relation

$$3\frac{\partial \varphi}{\partial J_1} + 2J_1\frac{\partial \varphi}{\partial J_2} + 3J_2\frac{\partial \varphi}{\partial J_3} = 0 \tag{2.4}$$

must hold in addition to (2.3). The characteristics of the homogeneous, linear partial differential equation (2.4) are given by the relations $dJ_1/3 = dJ_2/2J_1 = dJ_3/3J_2$, which furnish the independent integrals

$$J_2' = J_2 - \frac{1}{3} J_1^2 \quad \text{and} \quad J_3' = J_3 - J_1 J_2 + \frac{2}{9} J_1^3 . \tag{2.5}$$

The dilatancy function φ therefore is a function of these integrals, which are readily seen to be the invariants

$$J_2' = \epsilon_{ij}\epsilon_{ji} \quad \text{and} \quad J_3' = \epsilon_{ik}\epsilon_{kj}\epsilon_{ji} \tag{2.6}$$

of the velocity strain deviator

$$\epsilon_{ij} = e_{ij} - \tfrac{1}{3} e_{kk}\, \delta_{ij} . \tag{2.7}$$

This restriction on φ implies that if e_{ij}^* represents a dilatant state of velocity strain then so does $e_{ij}^* + e\delta_{ij}$ for an arbitrary scalar e.

To sum up: if $-p$ in (2.1) is to be identical with the mean normal stress, the stress tensor is given by

$$p_{ij} = -p\, \delta_{ij} + 2\mu\epsilon_{ij} + 2\alpha \frac{\partial\varphi}{\partial J_2'} \epsilon_{ij} + 3\alpha \frac{\partial\varphi}{\partial J_3'} (\epsilon_{ik}\epsilon_{kj} - \tfrac{1}{3}\epsilon_{lk}\epsilon_{kl}\, \delta_{ij}). \tag{2.8}$$

If the fluid is treated as incompressible, the constitutive equations (2.8) may be retained except that the pressure p must now be regarded as the reaction to the kinematical constraint expressed by the incompressibility condition

$$e_{kk} = J_1 = 0. \tag{2.9}$$

For an incompressible fluid, the deviator ϵ_{ij} in (2.8) is of course identical with the velocity strain e_{ij}.

Throughout most of this paper, we shall be concerned with incompressible fluids obeying a constitutive equation of the form (2.8), in particular with the fluid defined by

$$\varphi = \tfrac{1}{2} J_2' - k^2, \tag{2.10}$$

where k is a characteristic constant with the dimensions T^{-1}. Since this dilatancy function corresponds to the von Mises' yield function in the theory of plasticity, this fluid will be named after von Mises. If the "viscous stress tensor" is defined by

$$s_{ij} = p_{ij} + p\, \delta_{ij} , \tag{2.11}$$

where the symbols on the right have the same meaning as in (2.8), it follows from (2.8) and (2.10) that

$$s_{ij} = (2\mu + \alpha)\epsilon_{ij} . \tag{2.12}$$

This equation still contains the non-negative factor α, which can be eliminated as follows. Multiplying each side of (2.12) by itself, using the first equation (2.6) and changing the dummy subscripts, we obtain

$$s_{kl}s_{kl} = (2\mu + \alpha)^2 J_2' . \tag{2.13}$$

According to the statement made in connection with the constitutive equation (2.1), the coefficient α vanishes when the fluid is in a regular state, i.e. when $J_2' < 2k^2$ for the

von Mises fluid. In this case, $s_{ij} = 2\mu\epsilon_{ij}$ and therefore $J_2' = (s_{kl}s_{kl})/(4\mu^2)$. If, on the other hand, $J_2' = 2k^2$, Eq. (2.13) furnishes the value of $2\mu + \alpha$, which can be substituted into (2.12). Thus, the full constitutive equation of the von Mises fluid can be written as follows

$$\epsilon_{ij} = \begin{cases} s_{ij}/(2\mu) & \text{if} \quad s_{kl}s_{kl} < 8\mu^2k^2, \\ ks_{ij}/(s_{kl}s_{kl}/2)^{1/2} & \text{if} \quad s_{kl}s_{kl} \geq 8\mu^2k^2. \end{cases} \tag{2.14}$$

Note that (2.14) furnishes a unique velocity strain, for a given stress, but does not possess a unique inverse.

It will be assumed throughout this paper that all components of stress and velocity strain are continuous functions of position. Accordingly, α must vanish on the interfaces of regions of regular and dilatant behaviour of the fluid.

3. Rectilinear flows. The equations of motion for an incompressible fluid, in the absence of body forces, can be written

$$s_{ij,j} = p_{,i} + \rho u_j u_{i,j} + \rho\, \partial u_i/\partial t \qquad (i = 1, 2, 3), \tag{3.1}$$

and

$$u_{i,i} = 0. \tag{3.2}$$

Steady flows in which only one Cartesian velocity component, u_1 say, is different from zero are called rectilinear flows. Such flows are compatible with (3.2) if, and only if,

$$u_1 = u_1(x_2\ , x_3), \tag{3.3}$$

from which we deduce that the inertial terms in (3.1) are identically zero.

In this section we will consider possible rectilinear flows of a von Mises fluid in an infinitely long cylindrical pipe with generators that are parallel to the x_1 axis; a typical cross-section of the pipe will be denoted by B and its contour by C. We will only consider cross-sections which are simply connected.

If the fluid is in a regular state, (3.3) and the first equation (2.14) imply that

$$s_{12} = \mu\, \partial u_1/\partial x_2\ , \qquad s_{13} = \mu\, \partial u_1/\partial x_3 \quad \text{and} \quad s_{11} = s_{22} = s_{33} = s_{23} = 0. \tag{3.4}$$

Equations (3.1) and (3.4) then show that

$$\partial p/\partial x_1 = -\theta, \quad \text{and} \quad \partial p/\partial x_2 = \partial p/\partial x_3 = 0, \tag{3.5}$$

where θ is a prescribed constant which we can assume to be non-negative. From (2.14) (3.1), (3.3) and (3.5) we deduce that

$$\nabla^2 u_1 = \partial^2 u_1/\partial x_2^2 + \partial^2 u_1/\partial x_3^2 = -\theta/\mu, \tag{3.6}$$

if, and only if,

$$\nabla u_1 \cdot \nabla u_1 = (\partial u_1/\partial x_2)^2 + (\partial u_1/\partial x_3)^2 < 4k^2; \tag{3.7}$$

otherwise the fluid is in a dilatant state and

$$\nabla u_1 \cdot \nabla u_1 = 4k^2. \tag{3.8}$$

Equations (3.6) and (3.8) are to be solved, for given values of θ, μ and k, subject to the condition

$$u_1 = 0 \quad \text{on} \quad C, \tag{3.9}$$

together with the condition that for finite θ, u_1 and its first derivatives are continuous on the interfaces Γ of regions of regular and dilatant behavior.

If the fluid is in a dilatant state Eq. (3.8) and the condition that u_1 be continuous on Γ determines u_1 uniquely: Eq. (2.12), (3.1), (3.3) and the assumed continuity of stress imply that (3.4) and (3.5) hold with μ replaced by $\alpha'\mu$ where $\alpha' = 1 + \alpha/(2\mu)$ satisfies the equation

$$\frac{\partial}{\partial x_2}(\alpha' \, \partial u_1/\partial x_2) + \frac{\partial}{\partial x_3}(\alpha' \, \partial u_1/\partial x_3) = -\theta/\mu, \tag{3.10}$$

subject to the condition

$$\alpha' = 1 \quad \text{on} \quad \Gamma. \tag{3.11}$$

Equations (3.6) through (3.9) are identical to the system of equations which occur in the study of the torsion of a cylinder with simply connected cross-section which is made of an elastic, perfectly plastic solid satisfying the von Mises yield condition [2]. To the variables u_1 and θ in the torsion problem there corresponds the stress function and the twist per unit length.

It is well known that there exists a unique solution to (3.6), satisfying (3.9), of the form

$$u_1 = \theta/\mu F(x_2 \, , x_3), \tag{3.12}$$

where $F(x_2 \, , x_3)$ is analytic, and non-negative, in the region B bounded by C. From (3.7) and (3.12) we conclude that, for sufficiently small θ/μ, only Newtonian flow occurs in the pipe. It can be shown that, on increasing θ/μ, condition (3.7) will be first violated on the boundary C of the simply connected cross-section. Further increase of θ/μ causes the regions, in which the flow is in a dilatant state to spread from the boundary towards the interior of the cross-section. The mathematical problem, for an arbitrarily given cross-section, is extremely difficult to solve because the curves Γ are not known before-hand. However, even though the mathematical problem appears intractable, there exists a very interesting analogy first noted by Nadai, which enables one to deduce by experiment the distribution of u_1 . The mathematical difficulties and the analogy are fully discussed in Ref. [2]. The only problem that has been solved directly is that which corresponds to the flow in a circular pipe.

The most rewarding technique used to date in the torsion problem is the semi-inverse method of Sokolovsky [5]. In its application to the flow problem discussed here a solution to (3.6) is taken and, for a particular choice of θ/μ, the curve Γ along which (3.8) is satisfied is determined. The curve Γ can then be taken as a transition curve and the region bounded by this curve can be taken as the core in which Newtonian flow occurs. Outside Γ u_1 satisfies (3.8) which can readily be integrated to determine a curve on which $u_1 = 0$; this curve is taken as the contour C of the pipe.

As an example of the semi-inverse method of Sokolovsky we determine the contour C which produces a Newtonian flow with a velocity field given by

$$u_1 = (\theta A/4\mu)\{a_0 - (r/A^{1/2})^2 + a_1(r/A^{1/2})^3 \cos 3\phi\} \tag{3.13}$$

for some value, $\bar{\theta}_0$ say, of $(\theta A^{1/2}/2\mu k)$. In (3.13), (r, ϕ) are the polar co-ordinates of a point in the cross-section B, A is the area of B, and a_0 and a_1 are specified constants.

The shape of C and Γ are shown in Fig. 3. Of great interest in the pipe flow of a dilatant fluid is the relation between the volume flow

$$\bar{Q} = \iint_B u_1 \, dx_1 \, dx_2 \tag{3.14}$$

and the non-dimensional pressure gradient

$$\bar{\theta} = \theta A^{1/2}/2\mu k. \tag{3.15}$$

When $\bar{\theta} < \bar{\theta}_c$, the value of $\bar{\theta}$ at which dilatant flow first occurs,

$$\bar{Q}/2kA^{3/2} = m\bar{\theta}, \tag{3.16}$$

where m is a constant which depends upon the contour C. Both $\bar{\theta}_c$ and m are computed numerically. As $\bar{\theta} \to \infty$ the transition curves asymptotically approach the ridges OA_1, OA_2, OA_3 shown in Fig. 3. The limiting value Q_L of $\bar{Q}/(2kA^{3/2})$ is readily determined

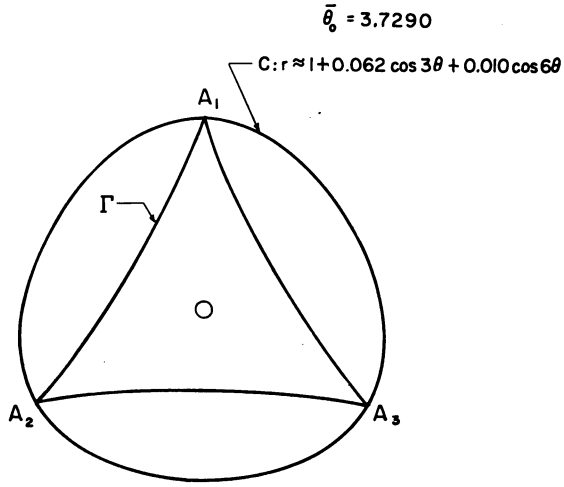

FIG. 3

numerically. The information obtained on the relation between $\bar{Q}/(2kA^{3/2})$ and $\bar{\theta}$ is exhibited in Fig. 4 and is compared with the known results for a circular pipe.

4. Boundary value problems. The principal problem which arises in the study of both Newtonian and dilatant fluids concerns the motion induced by rigid bodies moving through the fluid in a prescribed manner. For a Newtonian fluid it is tacitly assumed that the motions of the bodies can be prescribed arbitrarily. For a dilatant fluid the problem of determining the limits on the admissible motions of the bodies immersed in the fluid is fundamental to an understanding of its behavior. As a simple example we exhibit the shear flow between two infinite parallel plates moving at a constant relative velocity at a constant distance d apart. We specify that $u_1 = u_2 = u_3 = 0$ on the plane $x_2 = 0$ and that $p_{12} = \tau$, $p_{22} = -p$ and $p_{32} = 0$ on the plane $x_2 = d$, here τ and p are constants. The solution to the field equations subject to these conditions is

$$u_1 = \begin{cases} (\tau/\mu)x_2 & \text{if} \quad \bar{\theta} = |\, \tau/\mu k \,| \leq 2 \\ 2kx_2 \operatorname{sgn} \tau & \text{if} \quad \bar{\theta} \geq 2, \end{cases} \tag{4.1}$$

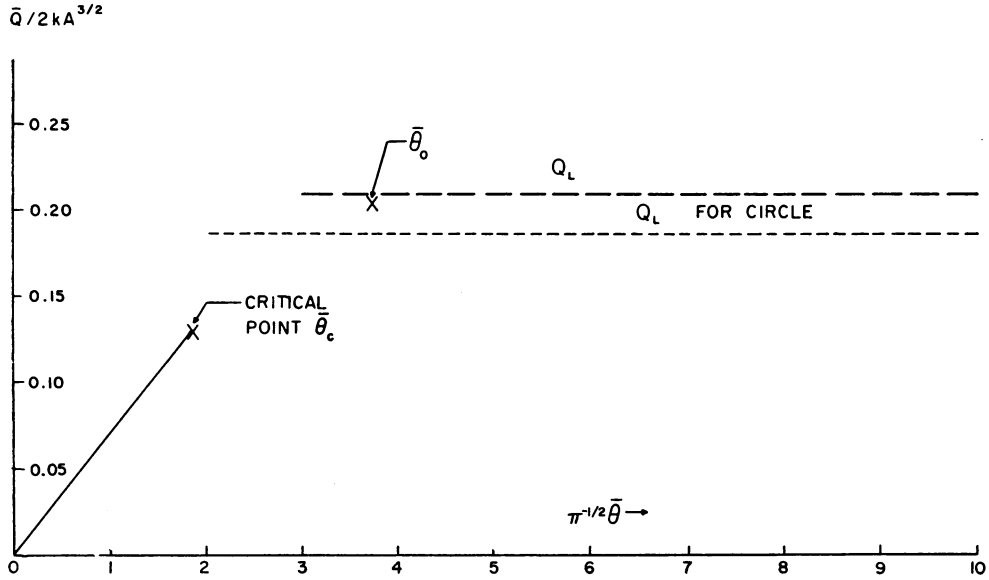

FIG. 4

while $u_2 = u_3 = p_{13} = p_{23} = 0$; $p_{11} = p_{22} = p_{33} = -p$ and $p_{12} = \tau$. From (4.1) we see that the velocity of the plate at $x_2 = d$ is restricted by $| u_1 | \leq 2kd$ for all τ; the limitation on the relative velocity between the two parallel plates is, of course, characteristic of the fluid discussed here.

In this section we consider the steady motion of a fluid, which satisfies the von Mises dilatancy condition,

$$\tfrac{1}{2}e_{ij}e_{ji} - k^2 \leq 0, \tag{4.2}$$

and is bounded by two closed, non-intersecting, regular rigid surfaces Γ_1 and Γ_2. The motion is caused by the relative motion of these surfaces. The condition that the motion shall be steady restricts both the forms of the surfaces Γ_1 and Γ_2 and their possible motions. In fact all admissible motions transform the surfaces into themselves. For instance, Γ_1 and Γ_2 could be bodies of revolution rotating with constant angular velocities about their common axis of revolution. We shall assume, for simplicity, that Γ_1 is at rest and that the velocities of Γ_2 are specified only to within a factor of proportionality. The condition that there be no relative motion between a particle of fluid at a point of the boundary and the boundary itself implies that

$$u_i = 0 \quad \text{on} \quad \Gamma_1 , \quad \text{and} \quad u_i = mu_i^0(x_p) \quad \text{on} \quad \Gamma_2 , \tag{4.3}$$

where $u_i^0(x_p)$ is a known function and m is a non-dimensional parameter. We seek the upper bounds on m imposed by the dilatant character of the fluid.

A Maximum principle for m. The equations expressing the conservation of momentum and mass for the steady flow of any fluid in the absence of body forces are

$$p_{ij,j} = \rho u_j u_{i,j} , \quad \text{and} \quad [\rho u_i]_{,i} = 0. \tag{4.4}$$

If we define the "Reynolds stresses" R_{ij} by

$$R_{ij} = p_{ij} - \rho u_i u_j , \qquad (=R_{ji}), \tag{4.5}$$

Eqs. (4.4) imply that

$$R_{ij,j} = 0. \tag{4.6}$$

A set of functions $R^*_{ij}(x_p)$ will be called a *statically admissible* Reynolds stress field if

(a) R^*_{ij} are of class C^1 in D, the region bounded by Γ_1 and Γ_2.

(b) $R^*_{ij} = R^*_{ji}$. (4.7)

(c) $R^*_{ij,j} = 0$.

The divergence theorem and (4.7) imply that if u_i is an arbitrary velocity field and e_{ij} the corresponding field of velocity strain

$$\iint_{\Gamma_1+\Gamma_2} u_i R^*_{ij} n_j \, d\Gamma = \iiint_D e_{ij} R^*_{ij} \, d\, D, \tag{4.8}$$

where n_j are the components of the normal pointing away from D on Γ_1 and Γ_2, $d\Gamma$ denotes an element of area of Γ_1 or Γ_2, and dD denotes an element of volume of D. If u_i satisfy conditions (4.3) then by the Schwartz inequality

$$e_{ij} R^*_{ij} \leq | e_{ij} e_{ij} |^{1/2} \, | R^*_{pq} R^*_{pq} |^{1/2}, \tag{4.9}$$

and Eq. (4.3) we can deduce from (4.8) that for a von Mises fluid

$$m \leq 2^{1/2} k \iiint_D | R^*_{pq} R^*_{pq} |^{1/2} \, d\, D \Big/ \iint_{\Gamma_1+\Gamma_2} u^0_i R^*_{ij} n_j \, d\Gamma. \tag{4.10}$$

The inequality (4.10) furnishes a series of upper bounds for m corresponding to a series of admissible functions R^*_{ij}. The aim, of course, is to minimize the right-hand side of (4.10) by a proper choice of R^*_{ij}.

As an example of the above maximum principle we consider the motion between two coaxial circular cylinders. The outer one, of radius d, is assumed to be at rest while the inner one, of radius b, rotates with a given angular velocity for which we wish to determined an upper bound. Letting the surface velocities u^0_i correspond to the unit angular velocity we may identify the factor in (4.10) with the magnitude ω of the bounding angular velocity.

We take the x_3 axis parallel to the generators of the cylinders and the origin of coordinates at a point on the axis of the cylinders. The functions R^*_{ij} we choose as follows

$$R^*_{11} = -\sin 2\theta/r^2, \; R^*_{22} = \sin 2\theta/r^2, \; R^*_{12} = \cos 2\theta/r^2, \text{ and } R^*_{33} = R^*_{32} = R^*_{31} = 0, \tag{4.11}$$

where (r, θ) are the cylindrical polar coordinates of a point in the fluid. The Reynolds stresses R^*_{ij} given by (4.11) possess the symmetry properties of polar symmetric flow. On $r = d$,

$$u_1 = u_2 = u_3 = 0, \tag{4.12}$$

on $r = b$

$$u_1 = -\omega b \sin \theta, \qquad u_2 = \omega b \cos \theta, \quad \text{and} \quad u_3 = 0.$$

By (4.10) (4.11) and (4.12) we deduce that

$$\text{Max} \, | \omega | \leq 2k \ln (d/b). \tag{4.13}$$

It can be shown that the upper bound on $|\omega|$ is exactly $2k \mid n(d/b)$; this result is, to a large extent, fortuitous.

The maximum principle formulated above is motivated by the limit analysis of Drucker, Greenberg, and Prager for an elastic-plastic body in plane strain [3]. In the same paper these authors presented a minimum principle which has proved powerful in limit analysis. However, as will be seen in what follows, the analogous minimum principle for a dilatant fluid is of limited application.

A set of functions $u_i^*(x_p)$ is called a kinematically admissible velocity field if

(a) u_i^* satisfy the condition of incompressibility in D,

(b) the corresponding velocity strain field e_{ij}^* satisfies the dilatancy condition (4.2) in D,

(c) u_i^* satisfy the boundary conditions (4.3) for some value, m_k say, of m.

The minimum principle can be expressed as follows. The upper limit on $|m|$ for a flow which is in a wholly dilatant state in D and for which the inertial terms can be neglected in the equations of motion is not less than the maximum values of $|m_k|$.

If the inertial terms are neglected, the momentum equations can be written

$$p_{ij,j} = 0. \tag{4.14}$$

By the divergence theorem, (2.12), (4.3) and (4.14) we deduce that

$$(m - m_k) \iint_{\Gamma_2} u_i^0 p_{ij} n_j \, d\Gamma = \iiint_D 2\mu\alpha'(e_{ij} - e_{ij}^*) e_{ij} \, dD. \tag{4.15}$$

By the Schwartz inequality and the properties of e_{ij}^* we have

$$e_{ij}e_{ij}^* \leq |e_{ij}e_{ij}|^{1/2} \, |e_{ij}^*e_{ij}^*|^{1/2} \leq (2)^{1/2}k \, |e_{ij}e_{ij}|. \tag{4.16}$$

If the fluid in D is in a wholly dilatant state, Eq. (4.16) and the conditions that μ, $\alpha' > 0$ imply that the right hand side of (4.15) is not negative for any admissible m_k. Since it is easily seen that

$$m \iint u_i^0 p_{ij} n_j \, d\Gamma \geq 0, \tag{4.17}$$

the minimum principle is established.

It should be noted that the minimum principle holds only if we know *a priori* that the boundary conditions are such as to make the flow wholly dilatant in D.

Uniqueness of creeping motion. The equations governing the creeping motion of the incompressible dilatant fluid discussed in Sec. 2 can be written

$$p_{ij,j} = 0, \tag{4.18}$$

and

$$u_{i,i} = 0, \tag{4.19}$$

where p_{ij} is given by (2.1).

Consider any finite, simply connected region D of the flow field with boundary B. Let us, at every point P of B, suppose that in each of the three coordinate directions either the component of the surface traction $\bar{T}$ or the component of the surface velocity $\bar{u}$

is known. Does this information determine u_i, e_{ij} and α uniquely in D? To investigate this, assume that u_i, e_{ij}^* and α^* also constitute a solution to the field equations satisfying conditions on B. Since at P

$$(u_i - u_i^*)(T_i - T_i^*) = 0, \tag{4.20}$$

then

$$\iint_B (u_i - u_i^*)(T_i - T_i^*)\, dB = \iint_B (u_i - u_i^*)(p_{ij} - p_{ij}^*)n_j\, dB = 0, \tag{4.21}$$

where n_j are the components of the unit outward normal to D at P. It is assumed that B is a regular surface, that u_i are C^1 (hence e_{ij} are C^0) and that p_{ij} are C^1 in D. Applying the divergence theorem to (4.21) and using (4.18) and (4.19) we obtain

$$2\mu \iiint_D (e_{ij} - e_{ij}^*)(e_{ij} - e_{ij}^*)\, d\, D + \iiint_D \alpha \frac{\partial \varphi}{\partial e_{ij}} (e_{ij} - e_{ij}^*)\, d\, D$$
$$+ \iiint_D \alpha^* \frac{\partial \varphi}{\partial e_{ij}^*} (e_{ij}^* - e_{ij})\, dD = 0. \tag{4.22}$$

We now restrict our consideration to dilatancy functions $\varphi(e_{ij})$ that are represented in velocity strain space by convex dilatancy surfaces. This condition implies that if e_{pq} is a state of velocity strain for which $\varphi(e_{pq}) = 0$, and if e_{pq}^* is any other attainable state of velocity strain (i.e. if it is represented by a point interior to, or lying on, the dilatancy surface) then

$$(e_{ij} - e_{ij}^*) \frac{\partial \varphi}{\partial e_{ij}} (e_{pq}) \geq 0. \tag{4.23}$$

The condition that $\alpha = 0$ if the fluid is in a regular state and Eq. (4.23) ensure that each term on the left-hand side of (4.22) is non-negative; from this we immediately deduce that

$$e_{ij} = e_{ij}^* \tag{4.24}$$

throughout D. From (4.24) we can conclude that u_i^* differs from u_i by at most a velocity field of a rigid-body motion. In any subregion D_N of D in which the fluid is in a regular state ($\alpha = 0$) the conditions $e_{ij} = e_{ij}^*$ and $u_i = u_i^*$ imply that

$$\frac{\partial}{\partial x_i}(p - p^*) = 0 \qquad (i = 1, 2, 3).. \tag{4.25}$$

From (4.24), (4.25) and (2.1) it follows that in D_N the total stress is uniquely determined to within a constant hydrostatic pressure. Of course if any point on the boundary of D_N coincides with a point of B at which the normal surface traction is specified then the components of p_i, are uniquely determined within D_N.

The flow between two parallel plates shows that the stress field is not uniquely determined when the fluid is in a dilatant state. As the region D we take the cube bounded by the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_1 = d$, $x_2 = d$ and $x_3 = d$. Instead of specifying $p_{12} = \tau$ we specify that $u_1 = 2kd$ on the plane $x_2 = d$. In addition on $x_1 = 0$ and $x_2 = d$ we specify that $p_{11} = -p$ and $u_2 = u_3 = 0$; on $x_3 = 0$ and $x_3 = d$ we specify that $p_{33} = -p$,

and $p_{23} = u_2 = 0$. The solution in D is given by (4.1) which imply that any value of $\tau \geq 2\mu k$ is compatible with the boundary conditions and the dilatant character of the fluid.

5. Unsteady shear flow. Several interesting, non-trivial, two-dimensional flows of an incompressible von Mises dilatant fluid can be found for which the non-linear terms in the equations of motion are identically zero. We will only consider such flows which have a mean flow in the x_1 direction and are such that u_1 is a function of x_2 and t only.

Flat plate with uniform suction. As a simple example we consider the steady flow in a region $x_2 \geq 0$ past an infinite plate situated in the plane $x_2 = 0$.

We specify that on

$$x_2 = 0, \qquad u_2 = -v_0(<0); \tag{5.1}$$

and as

$$x_2 \to \infty, \qquad u_1 \to U_\infty(>0),$$

where v_0 and U_∞ are prescribed constants.

If we define the non-dimensional variables

$$u = u_1/U_\infty, \qquad v = u_2/v_0, \qquad y = v_0 x_2/\nu \quad \text{and} \quad \epsilon = 2\,DU_\infty/v_0$$

where

$$D = k\nu/U_\infty^2, \tag{5.2}$$

it is readily shown that when $\epsilon \geq 1$ the fluid is in the regular state for all $y \geq 0$ and the velocity field is given by

$$u = 1 - e^{-y}, \qquad v \equiv -1. \tag{5.3}$$

When $0 < \epsilon \leq 1$ the fluid is dilatant for $0 \leq y \leq (1 - \epsilon)/\epsilon$ and regular for all $y \geq (1 - \epsilon)/\epsilon$. In the dilatant region

$$u = \epsilon y \quad \text{and} \quad v \equiv -1; \tag{5.4}$$

in the regular region

$$u = 1 - \epsilon \exp\{-[y - (1 - \epsilon)/\epsilon]\}, \quad \text{and} \quad v \equiv -1. \tag{5.5}$$

The shearing stress at the wall is simply $\rho v_0 U_\infty$ for all $\epsilon > 0$ and the pressure is constant in the whole space.

Suddenly accelerated plane wall. The motion produced in a Newtonian fluid when an infinite plane wall is moved impulsively from rest is not only important in itself but also in that it can be used, together with the Rayleigh analogy, to motivate the classical Prandtl boundary layer theory for more complicated flows [5]. For a dilatant fluid no discontinuities in the velocity components of a particle are mechanically admissible since such discontinuities imply infinite velocity strains which, in turn, violate the dilatant property of the fluid. However, in much the same spirit as Sokolovsky, we can use the solution obtained for a Newtonian fluid as a possible flow of a dilatant fluid in a regular state. The dilatant flow adjacent to, and the motion of the plane wall producing, such a flow can then be readily determined. In this section we present the flow corresponding

to the Newtonian motion produced when a plane wall is given an impulsive velocity at $t = 0$.

The equations governing the plane, unsteady rectilinear flow of the dilatant fluid are

$$\partial u_1/\partial t = \nu\, \partial^2 u_1/\partial x_2^2 \tag{5.6}$$

if, and only if,

$$|\, \partial u_1/\partial x_2 \,| < 2k \tag{5.7}$$

otherwise

$$|\, \partial u_1/\partial x_2 \,| \equiv 2k, \tag{5.8}$$

If α', which satisfies the equation

$$\partial u_1/\partial t = \nu \frac{\partial}{\partial x_2} (\alpha'\, \partial u_1/\partial x_2), \tag{5.9}$$

is not less than unity. We seek solutions to (5.6)-(5.8), satisfying certain prescribed initial and boundary conditions, which are of class C^1 in the (x_2, t) plane. The condition that the components of stress shall be continuous then implies that $\alpha' = 1$ on the transition curves.

The solution to (5.6) satisfying the conditions

$$t = 0: \quad u_1 = 0 \quad \text{for all} \quad x_2\,; \tag{5.10}$$

$$t > 0: \quad u_1 = U_0 \quad \text{for} \quad x_2 = 0; \quad u_1 = 0 \quad \text{for} \quad x_2 = \infty,$$

is well known [4] and is given by

$$u_1 = U_0 u(\eta), \tag{5.11}$$

where

$$\eta = x_2/2(\nu t)^{1/2} \tag{5.12}$$

and

$$u = 1 - 2\pi^{-1/2} \int_0^\eta \exp(-\xi^2)\, d\xi. \tag{5.13}$$

The integral in (5.12) defines the complementary error function which has been tabulated. It is convenient to define the non-dimensional variables

$$\tau = k^2 \nu t/U_0^2 (\geq 0), \quad \text{and} \quad y = k x_2/U_0 (\geq 0). \tag{5.14}$$

The curve Γ along which $|\, \partial u_1/\partial x_2 \,| = 2k$ is determined from (5.11)-(5.14) and is given by

$$y_\tau^2 = -2\tau \log(4\pi\tau) \quad \text{for} \quad y \geq 0 \quad \text{and} \quad 0 \leq \tau \leq \frac{1}{4\pi}. \tag{5.15}$$

The curve Γ and the curve $y = 0$ bound a finite domain D in the (y, τ) plane (Fig. 5). Since in the region N, exterior to D, the solution (5.11)-(5.13) provides a Newtonian flow for which $|\, \partial u_1/\partial x_2 \,| < 2k$ this flow will be taken as a possible regular flow of the dilatant fluid. The dilatant flow in the region D bordering the prescribed Newtonian

flow is completely determined by (5.8) and (5.9) and the conditions that u_1 and $\partial u_1/\partial x_2$ shall be continuous across Γ. The flow in D is given by

$$u = u_0(\tau) - 2y \quad \text{for} \quad 0 \leq y \leq [-2\tau \log 4\pi\tau]^{1/2}, \tag{5.16}$$

where

$$u_0(\tau) = 1 - 2\pi^{-1/2} \int_0^{\xi=[-\log 4\pi\tau]^{1/2}} \exp(-\xi^2)\, d\xi + 2[-2\tau \log(4\pi\tau)]^{1/2}. \tag{5.17}$$

By (5.8) and (5.9) and the requirement $\alpha' = 1$ on Γ we obtain

$$\alpha' - 1 = \frac{1}{2}\frac{du_0}{d\tau}\{[-2\tau \log(4\pi\tau)]^{1/2} - y\}. \tag{5.18}$$

From (5.16) we deduce that if, for $0 \leq \tau \leq 1/4\pi$, the plane wall $y = 0$ is moved with a velocity $U_0 u_0(\tau)$ the flow produced is given by (5.15) and (5.12). For all $\tau \geq 1/4\pi$ the flow is wholly Newtonian and corresponds to a plane wall moving with uniform

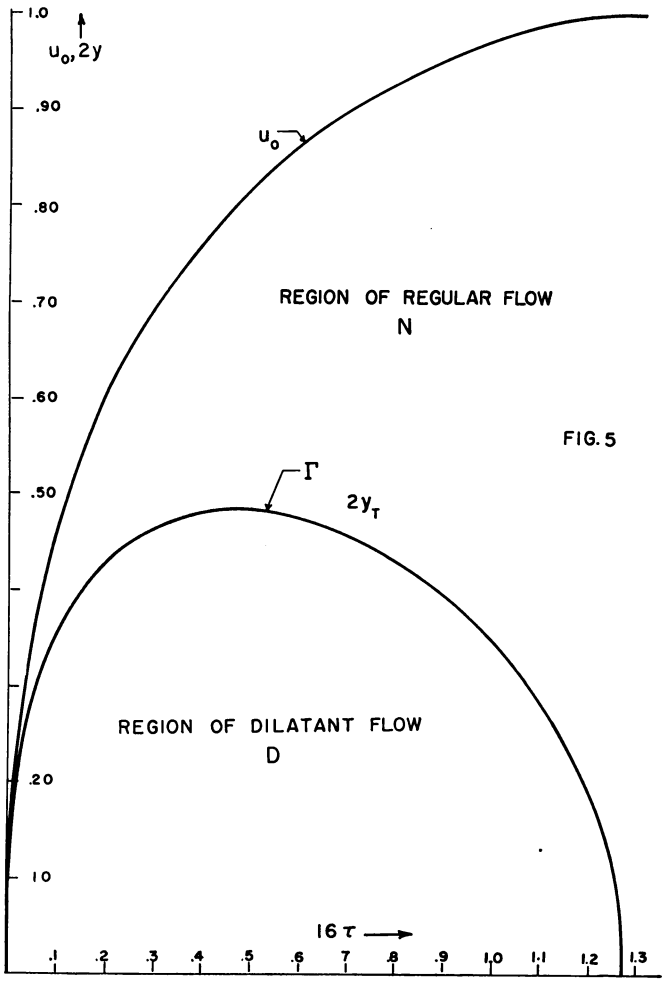

FIG. 5

velocity U_0. In Fig. 5 we exhibit $u_0(\tau)$ and the transition curve Γ for $0 \leq \tau \leq 1/4\pi$.

It should be noted that the flow produced by the motion of the wall given in Fig. 5 depends only upon the three dimensional constants k, ν and U_0. The only non-dimensional parameter which can be formed from these characteristic constants is

$$D = (k\nu/U_0^2). \tag{5.19}$$

From (5.14) and (5.15) we see that a dilatant layer of fluid exists for

$$0 \leq kt \leq \frac{4}{\pi} D^{-1}. \tag{5.20}$$

This layer is contained between the boundary wall and a regular flow region. It can be shown that the interface between the regular and dilatant state begins its motion towards the boundary wall when the acceleration of the wall has a stationary value.

Oscillating flat plate. In the example of the suddenly accelerated flat plate the only dimensional constant defined by the boundary and initial conditions was the maximum velocity of the plate, U_0. This constant, together with the characteristic constants ν and k of the fluid, defined the non-dimensional parameter D. In this section we consider a motion of the boundary which introduced two dimensional quantities; the maximum velocity U_0, and the frequency of the oscillation ω, of the plate.

It is convenient to introduce the non-dimensional variables

$$u = u_1/U_0, \qquad \tau = \omega t, \qquad y = (\omega/\nu)^{1/2} x_3 \quad \text{and} \quad \beta = 2\, D^{1/2}(k/\omega)^{1/2}, \tag{5.21}$$

where

$$D = k\nu/U_0^2. \tag{5.22}$$

In terms of the new variables the basic equations (5.6)-(5.9) become

$$\partial u/\partial \tau = \partial^2 u/\partial y^2 \tag{5.23}$$

if, and only if,

$$|\, \partial u/\partial y \,| < \beta; \tag{5.24}$$

otherwise

$$|\, \partial u/\partial y \,| = \beta \tag{5.25}$$

and

$$\partial u/\partial \tau = \frac{\partial}{\partial y} (\alpha' \, \partial u/\partial y) \tag{5.26}$$

if $\alpha' \geq 1$.

As the flow in the regular state we take the solution to (5.23) given by

$$u = e^{-y} \sin \eta \tag{5.27}$$

where

$$\eta = 2\tau - y. \tag{5.28}$$

The parallel flow given by (5.27) corresponds to the flow induced in a Newtonian fluid occupying the region $y \geq 0$, by an infinite flat plate moving with a velocity $U_0 \sin 2\omega t$. For the flow given by (5.27)

$$\partial u/\partial y = -2^{1/2} e^{-y} \sin (\eta + \pi/4). \tag{5.29}$$

Since the motion is periodic in time it can be determined from its behaviour for $0 \leq \tau \leq \pi$.

From (5.29) we see that the regular flow given by (5.27) is such that, at a fixed value of y,

$$| \partial u/\partial y | \leq (2)^{1/2} e^{-y} \leq (2)^{1/2} \quad \text{for all} \quad y \geq 0. \tag{5.30}$$

By (5.30) and (5.24) we deduce that when $\beta \geq (2)^{1/2}$ the dilatant fluid is in a wholly regular state and the motion is, of course, generated by a flat plate moving with a velocity $U_0 \sin 2\omega t$.

If a region of dilatant flow can exist for $\beta \leq (2)^{1/2}$ adjacent to the region N in the (x, τ) plane the dilatant layer is restricted to $0 \leq y \leq \log (2^{1/2}\beta^{-1})$ (Fig. 6). In order to

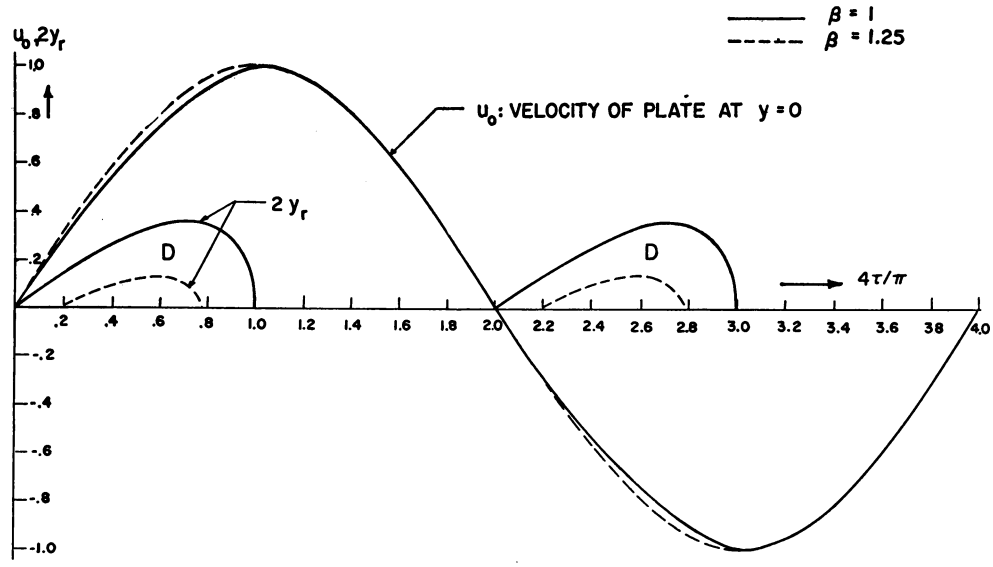

FIG. 6

determine the regions D in the (y, τ) plane in which the flow is dilatant it is sufficient to determine the regions for $0 \leq \tau \leq \pi$ in which $\partial u/\partial y = -\beta$ and $\alpha' \geq 1$; for if $y = y_T(t)$ represents the transition curve between the regular flow given by (6.27) and the dilatant flow given by $\partial u/\partial y = -\beta$ then $y = y_T(\tau + \pi/2)$ represents the transition curve between the regular flow and the dilatant flow given by $\partial u/\partial y = \beta$.

It is convenient to introduce the variables

$$\bar{y} = y + \log \beta, \ \bar{\tau} = \tau + \tfrac{1}{2} \log \beta, \ \bar{\eta} = 2\bar{\tau} - \bar{y} \text{ and } \bar{u}_0 = \exp(-\bar{y}_T) \sin \bar{\eta}_T + \bar{y}_T \tag{5.31}$$

in terms of which the transition curve along which $\partial u/\partial y = -\beta$ is given by

$$\exp(\bar{y}_T) = 2^{1/2} \sin [\bar{\eta}_T + \pi/4], \tag{5.32}$$

and the shear flow in D by

$$u = \beta[\bar{u}_0 - \bar{y}] \quad \text{and} \quad \alpha = 1 + (d\bar{u}_0/d\tau)[\bar{y}_T - \bar{y}], \tag{5.33}$$

where

$$\bar{u}_0 = \exp(-\bar{y}_T) \sin \eta_T + \bar{y}_T . \tag{5.34}$$

Since, when $\beta = 1$, $\bar{y} = y$ and $\bar{\tau} \equiv \tau$ the relations (5.31)-(5.34) provide the details of the flow in D for any admissible β in terms of the flow when $\beta = 1$. Although the curve given by (5.32) and the curve $y = 0$ form a closed contour in the (y, τ) plane for all $0 < \beta \leq (2)^{1/2}$ only for the range $1 \leq \beta \leq (2)^{1/2}$ is the flow in the region bounded by the contour such that $\alpha \geq 1$. Hence only when $\beta \geq 1$ can we find a motion of the flat plate at $y = 0$ which produces the regular flow given by (5.27).

When $1 \leq \beta \leq (2)^{1/2}$ there exists a dilatant layer of flow in which $\partial u/\partial y = -\beta$ for the time intervals

$$n\pi + \tfrac{1}{2}[\sin^{-1} \beta/2^{1/2} - \pi/4] \leq \tau \leq \tfrac{1}{2}[3\pi/4 - \sin^{-1} \beta/2^{1/2}] + n\pi, \quad n = 0, 1, 2, \cdots \tag{5.35}$$

and a dilatant layer of flow in which $\partial u/\partial y = \beta$ for the time intervals

$$(m + 1/2)\pi + \tfrac{1}{2}[\sin^{-1} \beta/2^{1/2} - \pi/4] \leq \tau \leq \tfrac{1}{2}[3\pi/4 - \sin^{-1} \beta/2^{1/2}] + (m + 1/2)\pi,$$
$$m = 0, 1, 2, \cdots ; \tag{5.36}$$

for all other times the flow is wholly regular. It should be noted that the flow is wholly regular for at least half of the period taken for a complete oscillation of the plate.

In Fig. 6 we exhibit the thicknesses of the dilatant layers y_T, and the motions of the flat plate at $y = 0$ which would produce them, as functions of time for two values of the parameter β. It can be shown that for all values of β at which dilatant layers occur the interfaces between the regular and dilatant states begin their motion towards the plane wall when the acceleration of the wall has a stationary value.

8. Conclusion. We have constructed a model of a dilatant fluid and have discussed certain general features of its flow. Since the interface between the dilatant region and the regular region is not known beforehand and since the Navier-stokes equations which govern the flow in the regular region defy, as yet, mathematical treatment, the direct problem of determining the flow past a given body seems intractable. By far the most powerful technique for obtaining a detailed description of a flow is the semi-inverse technique of constructing the dilatant flow adjacent to an *assumed* Newtonian flow; the boundaries producing such a flow are then determined. By this technique we have discussed the flow down pipes, and the flow adjacent to moving planes. The results indicate that dilatant fluids have many interesting properties.

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