A CLASS OF STABILITY CRITERIA FOR HILL'S EQUATION*

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The purpose of this note is to prove the following theorem: *A sufficient condition for the boundedness of all solutions of*

$$y'' + p(t)y = 0,$$

where $p(t)$ is an even, positive, differentiable function of period T, is that

$$k\pi \le \int_0^T [p(t)]^{1/2}\,dt - \tfrac{1}{4}\int_0^T \left|\frac{p'(t)}{p(t)}\right| dt \le \int_0^T [p(t)]^{1/2}\,dt + \tfrac{1}{4}\int_0^T \left|\frac{p'(t)}{p(t)}\right| dt \le (k+1)\pi$$

for some integer $k \ge 0$.

The proof of the theorem is based on a method developed in connection with a general analysis of the Sturm-Liouville spectrum [1]. There it is shown that the even and odd solutions of the differential equation, which are denoted by y_1 and y_2 respectively, can be represented as

$$y_1 = A_1(t)\cos\phi_1(t), \qquad y_2 = A_2(t)\sin\phi_2(t)$$

$$y_1' = -[p(t)]^{1/2}A_1(t)\sin\phi_1(t), \qquad y_2' = [p(t)]^{1/2}A_2(t)\cos\phi_2(t).$$

A direct calculation shows that the functions $\phi_i(t)$, $A_i(t)$ must satisfy the differential equations

$$\phi_1' = [p(t)]^{1/2} - \tfrac{1}{4}\frac{p'(t)}{p(t)}\sin 2\phi_1\,,$$

$$\phi_2' = [p(t)]^{1/2} + \tfrac{1}{4}\frac{p'(t)}{p(t)}\sin 2\phi_2\,,$$

$$A_1' = -A\frac{p'(t)}{2p(t)}(\sin\phi_1)^2,$$

$$A_2' = -A\frac{p'(t)}{2p(t)}(\cos\phi_2)^2,$$

and

$$A_1(0) = A_2(0) = 1, \quad \phi_1(0) = \phi_2(0) = 0.$$

If we consider a function $p(t)$ of the form

$$p(t) = \lambda + \Psi(t),$$

where λ is a parameter we obtain periodic solutions if and only if λ belongs to a discrete set of eigenvalues arranged in the following ascending sequence

$$-\infty < \lambda_0 < \lambda_1' \le \lambda_2' < \lambda_1 \le \lambda_2 < \lambda_3' \le \lambda_4' < \lambda_3 \le \lambda_4 \cdots$$

(see [2]). If λ is equal to some λ_i at least one solution of the equation has period T, but

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if λ is equal to some λ_i' at least one solution has period $2T$. One can easily show [1] that the λ_i must satisfy the conditions

$$\phi_1(T) = 2k\pi$$

if the corresponding periodic solution of the differential equation is even and

$$\phi_2(T) = 2k\pi$$

if the solution is odd. Similarly the λ_i' satisfy

$$\phi_1(T) = (2k + 1)\pi,$$

$$\phi_2(T) = (2k + 1)\pi,$$

corresponding to even and odd solutions respectively.

Whenever λ lies in one of the intervals

$$(\lambda_0 , \lambda_1'), \qquad (\lambda_2' , \lambda_1), \qquad (\lambda_2 , \lambda_3'), \cdots$$

both solutions of the differential equation are bounded for all real t[2]. This can only happen if both $\phi_i(T)$ satisfy the inequalities

$$k\pi \leq \phi_i(T) \leq (k + 1)\pi$$

for some integer $k \geq 0$. Since

$$\phi_i(T) = \int_0^T [p(t)]^{1/2}\, dt \mp \tfrac{1}{4} \int_0^T \frac{p'(t)}{p(t)} \sin 2\phi_i \, dt, \qquad \left\{ i = \begin{matrix} 1 \\ 2 \end{matrix} \right.$$

the conclusion of the theorem follows from elementary considerations. This theorem generalizes a stability criterion proved in an earlier publication [3].

References

[1] H. Hochstadt, *Asymptotic estimates for the Sturm-Liouville spectrum,* to be published, Comm. Pure and Appl. Math.

[2] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations,* McGraw-Hill, New York, 1955, Chapter 8.

[3] H. Hochstadt, *A stability criterion for Hill's equation,* to be published, Proc. Amer. Math. Soc.