and

$$\int_{-\infty}^{\infty} V^{-1}z^2 \, dx = \int_a^b V^{-1}z^2 \, dx \leqq \int_a^b V^{-1} \, dx/4(b - a).$$

Hence, the expression $[\cdots]$ of (17) is, for $g = z$, majorized by const. $(b - a)^{-3} \int_a^b V^{-1} \, dx$. It is clear that each z can be smoothed out so as to obtain a function g possessing continuous second derivatives of the type allowed in (17) and such that $[\cdots]$ has again the same majorant. Condition (16) now yields (17) and the proof of the Theorem is complete.

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AN UPPER BOUND ON NON-NEGATIVE TRANSIENT RESPONSES*

BY

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In a recent note [1], it was shown that, if the real-valued function $w(t)$ of the real variable t is zero for $t < 0$ and if its Laplace transform $W(s)$ is given by

$$W(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{N(s)}{D(s)}, \tag{1}$$

where $m \geq 2n$ and the real parts of the roots of the polynomial $D(s)$ are all non-positive,

*Received July 5, 1961.

then for $t > 0$

$$| w(t) | \leq \frac{| a_n | t^{m-n-1}}{(m - n - 1)!} + \frac{| a_{n-1} | t^{m-n}}{(m - n)!} + \cdots + \frac{| a_0 | t^{m-1}}{(m - 1)!}. \tag{2}$$

If we replace the restriction that $m \geq 2n$ by the condition that $w(t) \geq 0$, a similar inequality results and it has a very simple proof.

More specifically, *if $w(t)$ is a real-valued non-negative function of the real variable t and is zero for $t < 0$ and if its Laplace transform is given by* (1) *where $m > n$ and the roots of the polynomial $D(s)$ have non-positive real parts, then for $t > 0$*

$$w(t) \leq \frac{a_n t^{m-n-1}}{(m - n - 1)!} + \frac{a_{n-1} t^{m-n}}{(m - n)!} + \cdots + \frac{a_0 t^{m-1}}{(m - 1)!}. \tag{3}$$

To establish this result, let

$$F(s) = \frac{a_n s^n + \cdots + a_0}{s^m + \cdots + b_0} \cdot \frac{s^m + \cdots + b_0}{s^m}.$$

The corresponding inverse Laplace transform that is zero for $t < 0$ may be written as follows for $t > 0$.

$$\begin{aligned} f(t) &= \frac{a_n t^{m-n-1}}{(m - n - 1)!} + \cdots + \frac{a_0 t^{m-1}}{(m - 1)!} \\ &= w(t) + b_{m-1} \int_0^t w(x)\, dx + \cdots \\ &\quad + \frac{b_0}{(m - 1)!} \int_0^t (t - x)^{m-1} w(x)\, dx. \end{aligned} \tag{4}$$

Since the roots of $D(s)$ are real or appear in complex-conjugate pairs and their real parts are all non-positive, every b_i is non-negative. Thus, all terms on the right-hand side of (4) are non-negative so that (4) implies (3).

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AN APPLICATION OF THE EULER-MACLAURIN SUM FORMULA TO OPERATIONAL MATHEMATICS*

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Introduction. The use of operational methods is especially well adapted to the solution of various problems in applied mathematics. Thus, in problems on heat conduction one is often enabled to find special solutions of the heat balance equation suitable for large or small values of time. It is shown in the present study that the Euler-Maclaurin sum formula may be used in some cases to generate approximate solutions to the heat balance equation which give good results for all values of time, using only a limited number of terms.

*Received July 24, 1961.