THE SMALL-FIELD THEORY OF THE JOULE AND WIEDEMANN EFFECTS*

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Abstract. The simplest stress-strain-field relations valid for an isotropic, magnetostrictive medium are developed and applied to calculate the deformations of a wire in the case of the Joule effect (longitudinal magnetic field) and Wiedemann effect (helical magnetic field).

1. Introduction. Recently devices using the magnetostrictive effect, *i.e.*, the interaction between magnetic fields and mechanical deformations, have become increasingly important, particularly in the field of digital computers. The "twistor" is one example of such a device. Although it has been known for a long time, the magnetostrictive effect apparently has been studied very little from the point of view of the classical theory of the mechanics of continuous media. Two principal difficulties lie in the way of such a general study; the essential nonlinearity of the effect and the hysteretic behavior of most magnetic materials subjected to even moderately large fields. In the present investigation the second difficulty has been bypassed by restricting the analysis to reversible processes, i.e., to small fields.

Two effects in particular have received a large amount of attention in the literature. The first of these, the so-called 'Joule effect,' refers to the elongation (or contraction) of a magnetostrictive wire in a longitudinal magnetic field. The other, the Wiedemann effect, refers to the twist produced by the combined action of longitudinal and circumferential fields, the latter usually developed by a current in the wire.

In the following the most general stress-strain-field relations, governing the small deformations of an isotropic, magnetostrictive material in a reversible magnetic field are developed and applied to the Joule and Wiedemann effects.

2. General equations. The small motion of a continuous, magnetostrictive medium is governed by the equations

$$T_{ij,j} = \rho u_i'', \tag{2.1}$$

$$\epsilon_{ijk} H_{k,j} = D_i' + J_i \ , \tag{2.2}$$

$$\epsilon_{ijk} E_{k,j} = -B_i' \ . \tag{2.3}$$

Equation (2.1), in which T_{ij} is the stress tensor, ρ the mass density, and u_i the particle displacement, is the equation of motion for any continuous medium in the absence of body forces. Equations (2.2) and (2.3), in which E_i and H_i are the electric and magnetic field intensities, D_i the electric displacement, B_i the magnetic induction, and J_i the current density, are Maxwell's equations, which must be adjoined to the equation of motion in problems involving electromechanical interaction. Here and in the following, we use commas to denote differentiation with respect to the Cartesian coordinates $(x_1 \ , x_2 \ , x_3)$ and primes to denote differentiation with respect to the time. Repeated

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indices are to be summed over all values of the indices and the substitution tensor δ_{ij} and the alternating tensor ϵ_{ijk} are given by

$$\delta_{ij} = \begin{cases} 1, & \text{for} \quad i = j, \\ 0, & \text{for} \quad i \neq j, \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for } i, j, k \text{ in cyclic order,} \\ -1, & \text{for } i, j, k \text{ in anticyclic order,} \\ 0, & \text{otherwise.} \end{cases}$$

These equations are completed by the specification of appropriate boundary conditions and of suitable constitutive relations between T_{ij}, B_i, D_i, J_i, and u_i [or, more precisely, the strain $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$], H_i, E_i. Relations valid for the small, reversible deformations of an isotropic, magnetostrictive medium are derived in the following sections, using the principle of conservation of energy and certain symmetry considerations.

3. The conservation of energy. For any continuous medium the principle of the conservation of energy may be expressed in the mathematical form

$$U' = T_{ij}S'_{ij} + E_iD'_i + H_iB'_i + J_iE_i - q_{i,i} . \tag{3.1}$$

This equation is obtained by equating the rate of increase of internal plus kinetic energy in an arbitrary volume of the medium to the rate at which mechanical forces do work on the volume and the rate at which thermal and electromagnetic energy flow into the volume through its bounding surface. Both the equations of motion and Maxwell's equations have been used to reduce it to its present form. Furthermore, to avoid kinematic difficulties not pertinent to the present considerations, we have assumed that the deformations are small. This is not essential, but simplifies the algebra considerably.

If we assume, by analogy with classical thermodynamics, that the dissipative terms J_iE_i (ohmic heating) and $q_{i,i}$ (heat conduction) are given in terms of an entropy density function N and the temperature Θ, by

$$\Theta N' = J_iE_i - q_{i,i} , \tag{3.2}$$

then, in two special cases, we may show that Eq. (3.1) implies the existence of a so-called "energy potential," i.e., a scalar function F such that

$$T_{ij} = \partial F/\partial S_{ij} , \qquad B_i = -\partial F/\partial H_i .$$

First we assume that $N' = 0$, i.e., the medium is adiabatic. This might be closely approximated to by a medium with low electrical and thermal conductivity or in the consideration of times so short that electrical and thermal diffusion has not taken place to an appreciable extent (the dynamic problem). In this case, if U is assumed to be a function of the strain S_{ij}, the magnetic induction B_i, and the electric displacement D_i, the ordinary "chain rule" of partial differentiation gives

$$T_{ij} = \partial U/\partial S_{ij} , \qquad H_i = \partial U/\partial B_i , \qquad E_i = \partial U/\partial D_i \tag{3.3}$$

immediately from Eqs. (3.1) and (3.2). The assumed functional dependence of U corresponds to the assumption of reversibility or path independence. Only in this case may we obtain an energy potential. For example, in the case of a viscous fluid, U depends upon strain rate and Eq. (3.3) could not be obtained.

We next consider the isothermal case (Θ = const.), appropriate for times such that thermal and electrical equilibrium has been reached (the static problem). In this case, if we set $\Phi = U - \Theta N$ and again assume reversibility, we find

$$T_{ij} = \partial\Phi/\partial S_{ij} , \qquad E_i = \partial\Phi/\partial D_i , \qquad H_i = \partial\Phi/\partial B_i . \tag{3.4}$$

Obviously, the "derivation" given above is quite artificial. The strongest foundation it has is that of expediency. Henceforth we shall *assume* the existence of a scalar energy function $W(S_{ij}, H_i) + W'(D_i)$ such that

$$T_{ij} = \partial W/\partial S_{ij} , \qquad B_i = -\partial W/\partial H_i , \qquad E_i = \partial W'/\partial D_1 . \tag{3.5}$$

The function $W + W'$ is related to Φ by the expression

$$W + W' = \Phi - H_i B_i . \tag{3.6}$$

We have taken W to be a function of H_i rather then B_i simply as a matter of convenience in the treatment of static problems. In the case of a reversible process this choice is at our disposal.

4. The isotropic medium. The energy function W must, of course, be a proper scalar function, that is, for given strain tensor S_{ij} and field vector H_i, its value must be independent of the space coordinate system with respect to which their components are defined. We may ensure that this be so by taking suitable scalar functions of S_{ij} and H_i for the arguments of W. A convenient choice for these functions are the nine scalar or contracted products, given by

$$a_{(km)} = S_{ij} e_i^{(k)} e_j^{(m)},$$
$$b_{(k)} = H_i e_i^{(k)},$$

where the $e_i^{(k)}$'s are three orthogonal unit vectors and the parentheses indicate that the indices enclosed are not tensorial in character. We may imagine that these unit vectors specify the orientation of the body with respect to the coordinate system in which S_{ij} and H_i are defined. We then have

$$W = W[a_{(11)}, a_{(22)}, a_{(33)}, a_{(12)}, a_{(23)}, a_{(13)}, b_{(1)}, b_{(2)}, b_{(3)}]$$

as the most general possible functional form of W. Clearly the value of W for fixed S_{ij} and H_i depends upon the choice of $e_i^{(k)}$, as one would expect, for example, in an anisotropic medium. Usually one may restrict the above for a particular medium by taking advantage of symmetries present, i.e., invariance under certain special classes of unit vectors. In particular, an isotropic medium may be defined as one in which the value of W, for fixed S_{ij} and H_i, is completely independent of the choice of the unit vectors $e_i^{(k)}$. In this case W may be taken as a symmetric function of the three principal strains $S_{(k)}$ and a symmetric, even function of the projections of the field vector H_i on the three principal directions. The principal directions are given by three orthogonal unit vectors $u_i^{(k)}$ such that

$$S_{ij} u_i^{(k)} u_j^{(m)} = \begin{cases} S_{(k)} , & \text{for } k = m, \\ 0, & \text{for } k \neq m. \end{cases}$$

Thus we take the isotropic energy potential W in the form

$$W = W(I_1 , I_2 , I_3 , I_4 , I_5 , I_6), \tag{4.1}$$

where the proper scalar invariants I_k are given by

$$I_1 = S_{(1)} + S_{(2)} + S_{(3)} = S_{ii} ,$$
$$I_2 = S^2_{(1)} + S^2_{(2)} + S^2_{(3)} = S_{ij}S_{ij} ,$$
$$I_3 = S^3_{(1)} + S^3_{(2)} + S^3_{(3)} = S_{ij}S_{jk}S_{ki} ,$$
$$I_4 = S_{(1)}h^2_{(1)} + S_{(2)}h^2_{(2)} + S_{(3)}h^2_{(3)} = S_{ij}H_iH_j ,$$
$$I_5 = S^2_{(1)}h^2_{(1)} + S^2_{(2)}h^2_{(2)} + S^2_{(3)}h^2_{(3)} = S_{ij}S_{jk}H_iH_k ,$$
$$I_6 = h^2_{(1)} + h^2_{(2)} + h^2_{(3)} = H_iH_i ,$$

with $h_{(k)}$ the projection of H_i on the k-th principal axis. It is easy to show that specification of I_1 to I_6 determines $S_{(1)}$, $S_{(2)}$, $S_{(3)}$, $h^2_{(1)}$, $h^2_{(2)}$, $h^2_{(3)}$ uniquely. Obviously, the I_k's are symmetric functions, as required. They are also homogeneous in degree of S_{ij} and H_i , which is convenient for approximation. We now assume that W is an analytic function and expand it around $S_{ij} = H_i = 0$. We take $W = T_{ij} = B_i = 0$, for $S_{ij} = H_i = 0$ and retain terms only up to order S^2, SH^2, H^2 in W, consistent with our assumptions of small deformations and reversible fields. Then W takes the form

$$W = \tfrac{1}{2}(\lambda I_1^2 + 2GI_2) + (mI_1I_6 + 2nI_4) - \tfrac{1}{2}\mu I_6 , \qquad (4.2)$$

where λ and G are elastic constants, m and n magnetostrictive constants, and μ the permeability constant. This yields the constitutive relations

$$T_{ij} = \partial W/\partial S_{ij} = \lambda S_{kk}\ \delta_{ij} + 2GS_{ij} + mH_kH_k\ \delta_{ij} + 2nH_iH_j , \qquad (4.3)$$

$$B_i = -\partial W/\partial H_i = \mu H_i - 2(mS_{kk}\ \delta_{ij} + 2nS_{ij})H_j . \qquad (4.4)$$

Equations (2.1), (2.2), (2.3), (4.3), and (4.4) form a complete set of equations describing the small, reversible deformation of an isotropic, magnetostrictive medium. To illustrate their application, we consider two simple cases of static deformation, namely the so-called Joule and Wiedemann effects.

5. The Joule effect. A particularly simple case is the deformation of a magnetostrictive wire produced by a constant longitudinal magnetic field $H_z = H_0$. All the equations are then satisfied by assuming that the strains are constant and that the stresses are zero. The strains can then be calculated directly from Eq. (4.1). In cylindrical polar coordinates (r, θ, z) one finds

$$S_{rr} = S_{\theta\theta} = -[(1 - 2\nu)m - 2\nu n]H_0^2/E, \qquad (5.1)$$

$$S_{zz} = -[(1 - 2\nu)m + 2n]H_0^2/E, \qquad (5.2)$$

where E is Young's modulus and ν Poisson's ratio, such that

$$\lambda = \nu E/(1 + \nu)(1 - 2\nu), \qquad G = E/2(1 + \nu).$$

The components of magnetic induction are given in this case by

$$B_r = B_\theta = 0, \qquad (5.3)$$

$$B_z = \mu H_0 + 2[2(1 - 2\nu)m^2 + (1 - 2\nu)(m + 2n)^2 + 8\nu n^2]H_0^3/E. \qquad (5.4)$$

Since for real materials $0 < \nu < \frac{1}{2}$, Eq. (5.4) indicates that the magnetostrictive effect always augments the induction, independent of the signs of m and n, that is, for such a body the $B - H$ curve is always initially concave upward.

6. The current-carrying wire. Before discussing the effect of combined axial magnetic field and current, i.e., the Wiedemann effect, we consider the deformations produced by the current alone. The magnetic field produced by a constant longitudinal current density J_0 is given by

$$H_r = H_z = 0, \qquad H_\theta = J_0 r/2. \tag{6.1}$$

In this case the equation of motion, Eq. (2.1), assumes the vector form

$$\operatorname{grad} [(\lambda + 2G) \operatorname{div} \mathbf{u} + (m - n)H_\theta^2] - G \operatorname{curl} \operatorname{curl} \mathbf{u} = 0. \tag{6.2}$$

A particular solution of this equation may be found easily by assuming that

$$u_r = u_0(r), \qquad u_\theta = u_z = 0,$$

Then Eq. (6.2) is satisfied by setting

$$\operatorname{div} \mathbf{u} = \partial(ru_0)/r\, \partial r = -(m - n)H_\theta^2/(\lambda + 2G) + \text{const.},$$

giving

$$u_0(r) = -(m - n)r[H_\theta^2(r) - H_\theta^2(a)]/4(\lambda + 2G), \tag{6.3}$$

where one integration constant has been chosen to make $u_0(a) = 0$ and the other to make $u_0(0)$ finite. The above constitutes a complete solution in the case where the wire is constrained against both axial and radial motion. The radial tension required to keep the radius fixed is

$$(T_{rr})_a = (m + n)H_\theta^2(a)/2. \tag{6.4}$$

The axial constraining tension, on the other hand, is given by

$$T_{zz} = -\lambda(m - n)[2H_\theta^2(r) - H_\theta^2(a)]/2(\lambda + 2G) + mH_\theta^2(r), \tag{6.5}$$

and the total restraining tensile force by

$$T = 2\pi \int_0^a rT_{zz}(r)\, dr = \pi a^2 m H_\theta^2(a)/2. \tag{6.6}$$

If the distribution of axial stress at the ends of the wire differs from that given by Eq. (6.5), the state of strain and magnetic field will be perturbed, but presumably this perturbation dies away far from the ends, so that the field and displacement at the middle of a wire of sufficient length is given by Eqs. (6.1) and (6.3), provided the total load is equal to T.

To determine the displacements in the case of the free wire, we imagine radial and axial loads equal and opposite to those given by Eqs. (6.4) and (6.6) to be applied to the wire. We find that the strains

$$S_{rr} = S_{\theta\theta} = u/r = -[(1 - 2\nu)m + (1 - \nu)n]H_\theta^2(a)/2E, \tag{6.7}$$

$$S_{zz} = \partial w/\partial z = -[(1 - 2\nu)m - 2\nu n]H_\theta^2(a)/2E, \tag{6.8}$$

must be added to the strains $(S_{rr})_0 = \partial u_0(r)/\partial r$, $(S_{\theta\theta})_0 = u_0(r)/r$, $(S_{zz})_0 = 0$, to obtain the strain distribution in the free wire.

7. The Wiedemann effect. Now we turn our attention to the deformations produced by the combined action of the axial magnetic field $H_z = H_0$ and the circumferential magnetic field $H_\theta = J_0 r/2$ produced by the current density $J_z = J_0$. The extensional deformations S_{rr}, $S_{\theta\theta}$, S_{zz} in this case can be calculated by simply superimposing the solutions found in the previous sections, since H_θ and H_z appear in the equations for the extensional stresses and strains only as squares and sums of squares, so that these equations are linear in their squares. The only new equation is that involving the product, $H_\theta H_z$, i.e.,

$$T_{\theta z} = 2GS_{\theta z} + 2nH_\theta H_z, \tag{7.1}$$

where

$$S_{\theta z} = \tfrac{1}{2}\, \partial v/\partial z,$$

If we set

$$v(r, z) = \varphi rz, \tag{7.2}$$

where φ is the twist per unit length, we find that

$$\varphi = -nH_0 J_0/G, \tag{7 3}$$

makes $T_{\theta z}$ identically zero. Thus, a constant twist per unit length, proportional to the product of axial field and axial current, is produced by their combined action. In contrast to the previous cases, this effect depends upon the direction of field and current.

As before, the divergence of the magnetic induction, calculated from Eq. (4.4), vanishes. The average axial magnetic induction $\langle B_z \rangle$, defined by

$$\pi a^2 \langle B_z \rangle = 2\pi \int_0^a rB_z(r)\, dr,$$

is given by

$$\langle B_z \rangle / H_z = \mu + [2(1 - 2\nu)(m + n)^2 + (1 - 2\nu)m^2 + 2(1 + 2\nu)n^2][H_\theta^2(a) + 2H_0^2]/E. \tag{7.4}$$

Again the average axial induction is augmented by the magnetostrictive effects.

8. Summary of results. In the following we list the elongation, radius change, volume change, and magnetic induction in the case of the Joule effect, the current carrying wire, and the Wiedemann effect.

I. *The Joule Effect*: ($H_r = H_\theta = 0$, $H_z = H_0 =$ const.)

Elongation:

$$\Delta L/L = -[(1 - 2\nu)m + 2n]H_0^2/E,$$

Radius change:

$$\Delta a/a = -[(1 - 2\nu)m - 2\nu n]H_0^2/E,$$

Volume change:

$$\Delta V/V = -(1 - 2\nu)(3m + 2n)H_0^2/E,$$

Magnetic induction:

$$B_r = B_\theta = 0,$$

$$B_z/H_z = \mu + 2[2(1 - 2\nu)(m + n)^2 + (1 - 2\nu)m^2 + 2(1 - 2\nu)n^2]H_0^2/E,$$

II. *The Current-Carrying Wire*: $(H_r = H_z = 0, H_\theta = J_0 r/2)$

Elongation:

$$\Delta L/L = -[(1 - 2\nu)m - 2\nu n]H_\theta^2(a)/2E,$$

Radius change:

$$\Delta a/a = -[(1 - 2\nu)m + (1 - \nu)n]H_\theta^2(a)/2E,$$

Volume change:

$$\Delta V/V = -(1 - 2\nu)(3m + 2n)H_\theta^2(a)/2E,$$

III. *The Wiedemann Effect*:

Elongation:

$$\Delta L/L = -[(1 - 2\nu)m - 2\nu n]H_\theta^2(a)/2E - [(1 - 2\nu)m + 2n]H_0^2/E,$$

Radius change:

$$\Delta a/a = -[(1 - 2\nu)m + (1 - \nu)n]H_\theta^2(a)/2E - [(1 - 2\nu)m - 2\nu n]H_0^2/E,$$

Volume change:

$$\Delta V/V = -(1 - 2\nu)(3m + 2n)[H_\theta^2(a) + 2H_0^2]/2E,$$

Twist per unit length:

$$\varphi = nH_0J_0/G,$$

Magnetic induction (average):

$$\bar{B}_z/H_z = \mu + [2(1 - 2\nu)(m + n)^2 + (1 - 2\nu)m^2 + 2(1 + 2\nu)n^2][H_\theta^2(a) + 2H_0^2]/E.$$

9. Discussion of results. We have now obtained exact solutions describing the deformation of the simplest magnetostrictive material under the action of a longitudinal magnetic field (the Joule effect), a circumferential magnetic field, and a helical magnetic field (the Wiedemann effect). The importance of these solutions lies not in their precise forms, but in the fact that they are all derived from a single set of stress-strain-field relations. These relations in turn are derived from a single scalar energy potenital, involving five material constants, i.e., two elastic constants, the permeability constant, and two magnetostrictive constants.

No attempt is made in the present analysis to relate the theory to microscopic theories of magnetization or to generalize it to include irreversible processes. The object is simply to demonstrate that even the simplest admissible description of magnetostriction predicts both the Joule and Wiedemann effects.

In the case of the Joule effect, we have a constant longitudinal magnetic field and all equations are satisfied by assuming the stresses to be identically zero and the strains to be constant. The state of strain may be regarded as made up of two parts, a uniaxial extension (or compression) parallel to the applied field and uniform dilation proportional to the square of the magnitude of the magnetic field. This volume change is proportional to $3m + 2n$, where m and n are the two magnetostrictive constants. This combination of constants then is the volume magnetostrictive constant. If it vanishes, magnetic fields alone produce no volume change. This might be a useful approximation, since for many magnetostrictive materials the volume changes produced by the application of a magnetic field are considerably smaller than the extensions.

We note in passing that the stress-strain-field relations imply in general that at any point where the stresses vanish one of the principal axes of strain coincides with the direction of the magnetic field vector. In this case the strain consists of a uniform dilatation (such as that produced by hydrostatic pressure) and a simple extension (or compression) in the direction of the field. One might suppose that one could always assume that the stresses vanish, at least for problems where no surface forces are exerted on the body. In this case the stress-strain-field relations would yield six linear equations for the six components of the strain in terms of products and squares of magnetic field components. However, the strains are required also to satisfy certain additional conditions, the so-called "compatibility equations," which are necessary and sufficient conditions that the strains be those which a continuous body may undergo, i.e., that the strain be derivable from a continuous displacement vector. One may exhibit cases in which these conditions cannot be satisfied in the absence of stresses. In fact the current-carrying wire is such a case. In this case, while the surface of the wire is free of tractions, there are, nevertheless, nonzero stresses in its interior.

When the circumferential field produced by the current and the longitudinal field are superimposed (the Wiedemann effect), extensions equal to the sum of the extensions in the individual cases are produced in the axial and radial directions. Besides these strains, a torsional strain proportional to the product of axial and circumferential field strength is produced. The extensions produced by the longitudinal field and by the current tend to cancel one another, so that one would expect to observe smaller extensions in the case of the Wiedemann effect than in the case of the Joule effect.