QUARTERLY OF APPLIED MATHEMATICS

Vol. XX APRIL, 1962 No. 1

REMARKS ON THE INERTIA INSTABILITY OF A ROLLING MISSILE*

BY
MAGNUS TIDEMAN
Svenska Aeroplan Aktiebolaget, Linköping, Sweden

Summary. Lyapunov's second method is applied to the question of stability when dynamic cross coupling is considered. The main result is a condition on the maximal non-symmetry that gives stable performance for any constant or non-constant angular velocity in roll. Methods are outlined for the treatment of some related questions.

Introduction. For modern aircraft capable of high rolling velocities and for missiles the phenomenon of inertia coupling instability, predicted by W. H. Phillips [2], is very important. The published theoretical investigations seem to treat only the case of a constant rolling velocity and to assume constant coefficients in the linearized equations of motion. These assumptions make it possible to use the theory of linear differential equations with constant coefficients.

The object of this paper is to show that the use of Ljapunov's second method and the theory of quadratic forms offers possibilities of extending the study to non-constant coefficients and velocity. When a missile has a certain degree of symmetry, inertia coupling will never give rise to instability. The main part of the study is devoted to criteria in this direction. We also indicate methods for estimating allowed rolling velocities, when the missile is not sufficiently symmetric. In order to prove that the results are reasonable, they are compared with those for a constant rolling velocity.

In Ref. [2] it is found that for given values of the natural dampings of the oscillations in pitch and yaw, the ratio of the natural frequencies has to be in a certain interval around 1, if the motion is to be stable for all constant values of the rolling velocity. The results are generalized to non-constant velocities. We also discuss the case when frequency and damping are changed by simple controlling devices.

1. Equations of motion.

1.1. Let xyz be a system fixed in the missile and assume that the velocity has small components $v\alpha_z$ and $v\alpha_y$ in the y- and z-directions, $v > 0$ being the velocity in the x-direction. Let J_x, J_y, J_z denote the moments of inertia and ω_x, ω_y, ω_z the angular velocities about the axes x, y, z. These axes are assumed to be principal inertia axes and to form the reference system for the aerodynamic coefficients c_y, c_z, e_y, e_z, f_y, f_z, which appear in Eq. (1.1) below. The mass will be denoted by m.

*Received April 14, 1961.

If we apply the equations of forces and moments in the y- and z-directions, we get the linearized equations:

$$\begin{aligned}
&\frac{d}{dt}(mv\alpha_z) + m(\omega_z v - \omega_x v\alpha_y) = -c_z\alpha_z ,\\
&\frac{d}{dt}(mv\alpha_y) + m(-\omega_y v + \omega_x v\alpha_z) = -c_y\alpha_y ,\\
&\frac{d}{dt}(J_y\omega_y) - \omega_x\omega_z(J_z - J_x) = -e_y\alpha_y - f_y\omega_y ,\\
&\frac{d}{dt}(J_z\omega_z) + \omega_x\omega_y(J_y - J_x) = e_z\alpha_z - f_z\omega_z ,
\end{aligned} \tag{1.1}$$

where the aerodynamic coefficients are assumed to be positive.

In order to deduce these equations we follow Ref. [1], where we use Tables I and II of the aerodynamic forces and moments. From Table I we have used Item 1 "forces due to angles of attack and yaw", from Table II Items 1 and 2 "moments due to angles of attack and yaw", and "moments due to pitching and yawing angular velocities". Items 6 and 7 of both tables are of no interest as regards stability. "Magnus pitching and yawing moments" and "moments due to rolling combined with pitching and yawing angular velocities" are assumed to have small effects and will not be considered until Sec. 7, where they are briefly discussed. Other forces and moments in the tables do not change the form of our equations, but would only change the numerical values of the coefficients. For simplicity they are omitted.

1.2. In the first attack we assume that the mass, the velocity v, the moments of inertia and the aerodynamic coefficients are all constants. When this assumption is relaxed, there are some modifications to be discussed in Sec. 6.

1.3. No equations have been used for the x-direction. This gap is filled by allowing v to be a variable (see Sec. 6 for modifications, when v is not a constant) and ω_x to be any quantity (provided that the mathematical operations performed have a meaning). This point of view seems to be of interest for a guided missile, where ω_x depends on the imposed manoeuvres.

Many investigations have been carried out for steady rolling. An important question is therefore whether consideration of a variable ω_x causes considerable changes of stability criteria. This equation is touched upon in Sec. 5.

It seems that for many questions it is sufficient to consider only the case of a constant rolling velocity.

1.4. Let $x_1 = mv\alpha_y$, $x_2 = J_y\omega_y$, $x_3 = mv\alpha_z$, $x_4 = J_z\omega_z$ form the vector X and let a dot denote differentiation with respect to t. The Eq. (1.1.) can be written as

$$X^{\cdot} + AX = 0, \tag{1.2}$$

where the matrix A has the form

$$\begin{bmatrix} C_1 & -J_1 & \omega_x & 0 \\ E_1 & F_1 & 0 & -K_1\omega_x \\ -\omega_x & 0 & C_2 & J_2 \\ 0 & K_2\omega_x & -E_2 & F_2 \end{bmatrix}. \tag{1.3}$$

with positive coefficients C, F, E, J, K connected with the coefficients of (1.1) by

$$\left.\begin{array}{llll} C_1 = c_y(mv)^{-1}, & C_2 = c_z(mv)^{-1}, & F_1 = f_yJ_y^{-1}, & F_2 = f_zJ_z^{-1} \\ E_1 = e_y(mv)^{-1}, & E_2 = e_z(mv)^{-1}, & J_1 = mvJ_y^{-1}, & J_2 = mvJ_z^{-1} \\ K_1 = (J_z - J_x)J_z^{-1} \text{ and } K_2 = (J_y - J_x)J_y^{-1} & & & \end{array}\right\}. \tag{1.4}$$

In the case of symmetry $C_1 = C_2$; $F_1 = F_2$; $E_1 = E_2$; $J_1 = J_2$; $K_1 = K_2$. The assumptions of Sec. 1.2 imply that A is constant.

2. On the mathematical tools for the investigation.

2.1. In what follows Q_1 and Q_2 denote quadratic forms in x_1, x_2, x_3, x_4, but also the symmetric matrices generating those forms. The form Q_1 is always assumed to be positive definite, i.e. a positive number q shall exist such that Q_1 is larger than $q(x_1^2 + \cdots + x_4^2)$ for all values of $X \neq 0$. The matrix Q_1 is constant, except in Sec. 7.

2.2. When the form Q_1 is given, Q_2 is calculated from

$$Q_2 = \frac{-d}{dt} Q_1 \tag{2.1}$$

using (1.2). If both Q_1 and Q_2 are positive definite, Eq. (2.1) obviously implies that X tends to zero when t tends to infinity. This is true for any initial condition on X. Consider for instance

$$Q_1 = x_1^2 + x_3^2 + q(K_2x_2^2 + K_1x_4^2), \tag{2.2}$$

where q is a positive parameter. Then

$$Q_2 = 2C_1x_1^2 + 2C_2x_3^2 + 2q(F_1K_2x_2^2 + F_2K_1x_4^2) + 2(qE_1K_2 - J_1)x_1x_2 + 2(J_2 - qE_2K_1)x_3x_4 . \tag{2.3}$$

If there is a positive number q such that (2.3) defines a positive definite form, then (1.2) and hence (1.1) is asymptotically stable for all finite functions ω_x . The choice (2.2) of Q_1 may seem to be a very special one, but we will prove that it is the only Q_1 such that Q_2 does not depend on ω_x . Since we shall restrict ourselves to quadratic forms Q_1 for "Ljapunov functions" it is natural to study (2.3).

The particular case, including that of symmetry, when $J_1E_1^{-1}K_2^{-1} = J_2E_2^{-1}K_1^{-1}(=q)$ is obvious. The question to be answered is, how much deviation from this strict equality (symmetry) can be allowed if (1.1) is to be stable?

2.3. The necessary and sufficient condition for a form Q_2 to be positive definite is that the characteristic values of the matrice Q_2 are all positive. Let these values be $s_1, \cdots, s_4$ which are all real since Q_2 is symmetric. An orthogonal transformation exists, which brings Q_2 into

$$Q_2 = s_1y_1^2 + s_2y_2^2 + s_3y_3^2 + s_4y_4^2 . \tag{2.4}$$

Assume that Q_2 is the sum of two forms Q_{21} and Q_{22} , and that s, s' and s'' are the smallest characteristic values of the corresponding symmetric matrices. From (2.4) one easily deduces

$$s \geq s' + s''. \tag{2.5}$$

2.4. Let A and B be symmetric matrices, $B \neq 0$, and s be a real number. Then the zeros ω_x of $| s + A + \omega_x B |$ are real. From (2.5) it follows that if we want Q_2, defined by (2.1), to be positive definite for all ω_x, it is necessary to choose Q_1 to make Q_2 independent of ω_x. Simple calculations show that the only possibility is (2.2) when $K_1K_2 \neq 1$. We have $K_1K_2 = (J_y - J_x)(J_z - J_x)(J_yJ_z)^{-1}$ smaller than 1, but also close to 1. It can be of interest to note the general form of the matrix Q_1 for which Q_2 is independent of ω_x in the case $K_1K_2 = 1$, namely

$$\begin{bmatrix} 1 & K\epsilon & 0 & \delta \\ K\epsilon & qK_2 & K\delta & 0 \\ 0 & K\delta & 1 & -\epsilon \\ \delta & 0 & -\epsilon & qK_1 \end{bmatrix}, \qquad K = K_2 = K_1^{-1}. \tag{2.6}$$

The numbers ϵ, δ, q must satisfy $\epsilon^2 + \delta^2 < q/K$ to give a positive definite form Q_1.

2.5. We make a remark on the case of time-dependent Q_1 and Q_2, connected through (2.1). Here positive definitness is not quite sufficient to secure stability of (1.2). If the characteristic values of Q_1 are allowed to tend to zero when t tends to infinity, we cannot conclude that X has the limit zero, even if Q_1 has. We shall in this case (Sec. 7) request that there are time-independent, positive definite forms Q_{ij} ; $i, j = 1, 2$, such that $Q_{i1} \leq Q_i \leq Q_{i2}$; $i = 1, 2$. This remark is more for theoretical completeness than of practical value for the applications, where we can safely assume Q_i to behave properly.

3. Condition for stability for any rolling velocity.

3.1. We return to (2.3). The matrix Q_2 has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with square matrices A, B of order 2. The secular equation for Q_2 reduces to the two second order equations for A and B. With $2C_1 = a$, $2C_2 = b$, $2F_1K_2 = c$, $2F_2K_1 = d$, $E_1K_2 = e$, $E_2K_1 = f$ and $J_1 = g$, $J_2 = h$ as temporary notations these equations in s are

$$(qe - g)^2 = (a - s)(qd - s), \tag{3.1}$$

$$(gf - h)^2 = (b - s)(qc - s). \tag{3.2}$$

A necessary and sufficient condition for all the roots of (3.1) and (3.2) to be positive is that q satisfy the inequalities

$$(qe - g)^2 \leq qad \quad \text{and} \quad (qf - h)^2 \leq qbc.$$

The first inequality is satisfied if $q_1 < q < q_2$, the second one if $q_3 < q < q_4$, where

$$q_{1,2} = (2e^2)^{-1}\{2ge + ad \mp [(ad)^2 + 4egad]^{1/2}\} \equiv \delta_1 \mp \delta_2 ,$$

$$q_{3,4} = (2f^2)^{-1}\{2hf + bc \mp [(bc)^2 + 4fhbc]^{1/2}\} = \delta_3 \mp \delta_4 .$$

The two intervals for q have a common part if and only if $| \delta_1 - \delta_3 | < \delta_2 + \delta_4$. This last inequality can be transformed into the following form, where we return to the notations of (1.3).

$$\begin{aligned} D^2 &\equiv [(K_1J_1J_2^{-1})^{1/2}(C_2F_2 + J_2E_2)^{1/2} - (K_2J_2J_1^{-1})^{1/2}(C_1F_1 + J_1E_1)^{1/2}]^2 \\ &< [(K_1J_1J_2^{-1}C_2F_2)^{1/2} + (K_2J_2J_1^{-1}C_1F_1)^{1/2}]^2 \equiv \phi_1 . \end{aligned} \tag{3.3}$$

The physical meaning of (3.3) is hidden behind the unusual symbols, which have been used for formal reasons. In Sec. 5 we shall give an interpretation in natural frequency, damping factor and time constants.

Theorem 1. The system (1.1) is stable for any (even a non-constant) rolling velocity if the inequality (3.3) is satisfied.

3.2. We have found in Theorem 1 a sufficient condition on the allowed asymmetry, if our system is to be stable. The system might be stable even if the condition is violated and it can be questioned how sharp the result is. We shall therefore consider the case of constant rolling velocity and derive an inequality similar to (3.3) in 3.4. This result is necessary and sufficient, and by comparing it with (3.3) the usefulness of Theorem 1 can be estimated. Section 2.4 indicates that the theorem is quite sharp.

3.3. The Eqs. (3.1), (3.2) can be used to estimate the "degree of stability", i.e. how fast X tends to zero. Let s be the smallest number among the four roots of these equations, r the maximum of 1, K_1q, K_2q and let $v = sr^{-1}$. Then

$$Q_1(t) \leq Q_1(0)e^{-vt} \tag{3.4}$$

since

$$Q_2 = \frac{d}{dt}Q_1 \leq -s(x_1^2 + \cdots + x_2^4) \leq -vQ_1$$

according to (2.4).

3.4. For a constant value Ω of the rolling velocity ω_x the system (1.2) is linear and has constant coefficients. The stability is governed by the real parts of the zeros s_i of the determinant

$$| A + s | \equiv s^4 + a_3(\Omega)s^3 + a_2(\Omega)s^2 + a_1(\Omega)s + a_0(\Omega).$$

The system is stable if and only if all the real parts are negative. A necessary condition is that the coefficients $a_0 \cdots a_3$ be positive. Consider in particular

$$\begin{aligned} a_0(\Omega) &= b_0 + b_1\Omega^2 + K_1K_2\Omega^4 \quad \text{with} \\ b_0 &= (C_1F_1 + E_1J_1)(C_2F_2 + E_2J_2) \quad \text{and} \\ b_1 &= C_1C_2K_1K_2 + F_1F_2 - J_1E_2K_1 - J_2E_1K_2 \,. \end{aligned}$$

It holds that a_0 is positive for all Ω if and only if $b_1 > 0$ or $b_1^2 < 4K_1K_2b_0$. The last inequality can be transformed into

$$D^2 < (C_2K_1J_1J_2^{-1} + F_1)(C_1K_2J_2J_1^{-1} + F_2) \equiv \phi_2 \,, \tag{3.5}$$

where D is defined in (3.3). If b_1 is positive (3.5) is satisfied. An application of Routh's test proves that condition (3.5) is also sufficient for stability.

Theorem 2. The system (1.2) is stable for all constant rolling velocities if and only if (3.5) is satisfied.

The inequalities (3.3) and (3.5) have almost the same form, their right members differing by

$$\phi_2 - \phi_1 = [(C_1C_2K_1K_2)^{1/2} - (F_1F_2)^{1/2}]^2. \tag{3.6}$$

The relations (3.3), (3.5) and (3.6) are difficult to discuss in their present form. We shall return to them in connection with a physical interpretation in Sec. 5.

4. Estimation of admissible rolling velocities in non-symmetric cases.

4.1. If (3.5) is violated we cannot have stable performance for all constant values Ω of ω_x. From 3.4 it follows that we have instability if

$$\Omega_1 \le |\Omega| \le \Omega_2, \quad \text{where} \quad \Omega_{1,2}^2 = \tfrac{1}{2}K_1^{-1}K_2^{-1}[-b_1 \mp (b_1^2 - 4K_1K_2b_0)^{1/2}]. \tag{4.1}$$

See Ref. [2]. The numbers $\Omega_{1,2}$ are the zeros of $a_0(\Omega)$.

The natural generalization to non-constant ω_x is to ask for upper bounds ω_0 on $|\omega_x|$ if (1.2) is to be stable for all ω_x of modulus smaller than ω_0. When (3.5) does not hold we find in Ω_1 an upper bound for ω_0, and we will first search for estimates in the other direction.

4.2. We consider

$$Q_1 = S_1x_1^2 + S_2x_2^2 + S_3x_3^2 + S_4x_4^2, \quad \text{all} \quad S_i \quad \text{positive.} \tag{4.2}$$

The corresponding matrix Q_2 is given by

$$\begin{pmatrix} 2S_1C_1 & E_1S_2 - J_1S_1 & \omega_x(S_1 - S_3) & 0 \\ E_1S_2 - J_1S_1 & 2S_2F_1 & 0 & \omega_x(K_2S_4 - K_1S_2) \\ \omega_x(S_1 - S_3) & 0 & 2S_3C_2 & J_2S_3 - E_2S_4 \\ 0 & \omega_x(K_2S_4 - K_1S_2) & J_2S_3 - E_2S_4 & 2S_4F_2 \end{pmatrix}. \tag{4.3}$$

The characteristic numbers of Q_2 are real. It follows that they are positive for all $|\omega_x| < \omega_0$, if they are positive when $\omega_x = 0$ and if the determinant of (4.3) is positive when $|\omega_x| \le \omega_0$. Any particular choice of Q_1 will in this way give an estimate of ω_0. Even if Q_1 is restricted according to (4.2) it is not easy to find the best choice of Q_1. We will be content with some special choices which give rather good results (see Sec. 5.4) with a small amount of computation.

We can introduce zeros for some of the elements in (4.3) and reduce the computations to matrices of order 2 if we choose

$$S_1 = J_2K_2E_1, \quad S_2 = J_1J_2K_2, \quad S_3 = J_1K_1E_2, \quad S_4 = J_1J_2K_1 \quad \text{or} \tag{4.4}$$

$$S_1 = S_3 = E_1E_2, \quad S_2 = E_2J_1, \quad S_4 = E_1J_2. \tag{4.5}$$

These choices of Q_1 yield the estimates

$$\omega_0^2 \ge 4C_1C_2J_1J_2K_1K_2E_1E_2(J_1K_1E_2 - J_2K_2E_1)^{-2}, \tag{4.6}$$

$$\omega_0^2 \ge 4J_1J_2F_1F_2E_1E_2(J_1K_1E_2 - J_2K_2E_1)^{-2}, \tag{4.7}$$

which are discussed in Sec. 5.

4.3. The notations

$$E_1J_1 + C_1F_1 = \Omega_y^2 \quad \text{and} \quad E_2J_2 + C_2F_2 = \Omega_z^2 \tag{4.8}$$

are introduced for later use. Assume for simplicity that

$$K_1 = K_2 = 1 \quad \text{and} \quad J_1 = J_2 (= J) \tag{4.9}$$

are good approximations. Let in (4.3)

$$S_4 = S_2 = J^2, \qquad S_1 = \Omega_y^2, \qquad S_3 = \Omega_z^2.$$

It follows that

$$\omega_0^2 \geq C_1C_2(4\Omega_y^2 - C_1F_1)(4\Omega_z^2 - C_2F_2)(\Omega_y^2 - \Omega_z^2)^{-2}. \tag{4.10}$$

In a similar way one proves that

$$\omega_0^2 \geq F_1F_2(A\Omega_y^2 - C_1F_1)(4\Omega_y^2 - C_2F_2)(\Omega_y^2 - \Omega_z^2)^{-2}. \tag{4.11}$$

4.4. The number ω_0 can also be estimated with the aid of (2.5). We will here use this method to prove that if (3.3) is not true, when (3.5) holds, instability will occur only if very high rolling velocities are allowed. The matrix Q_1 is considered to have the form (2.6), the number K being equal to $(1 + K_1)(1 + K_2)^{-1}$. The corresponding Q_2 has the form $Q_{21} + Q_{22}$, where Q_{22} depends on ω_x, but Q_{21} is independent of ω_x. The characteristic values of Q_{22} have the same modulus and

$$s'' = -|\,\omega_x\,|\,(\epsilon^2 + \delta^2)^{1/2}(1 - K_1K_2)(1 + K_2)^{-1}. \tag{4.12}$$

Assume that (3.3) is turned into equality. In that case there is a value q_0 of q, such that (3.1) and (3.2) both have $s = 0$ as their smallest solution. For this value of q, values of ϵ and δ exist such that the characteristic numbers of Q_{21} are larger than $M(\epsilon^2 + \delta^2)^{1/2}$, where M depends on C_1, C_2 $\cdots$. For sufficiently small values of ϵ and δ, M can be chosen to be positive and independent of ϵ, δ. An application of (2.5) using this and (4.12) proves that

$$\omega_0 \geq M(1 + K_2)(1 - K_1K_2)^{-1}. \tag{4.13}$$

When K_1 and K_2 are close to 1, high rolling velocities are hence needed to give instability. This is also true by continuity if (3.3) is violated with a small difference between the left and right members.

The result does not contradict the unconditional stability for high constant rolling velocities, since we have not assumed constant ω_x.

5. An interpretation.

5.1. Let Ω_y and ζ_y denote the natural frequency and damping factor for motions in the y-direction, which are found from the equations for x_1 and x_2 in (1.2), when $\omega_x = 0$

$$\Omega_y^2 = J_1E_1 + C_1F_1 \quad \text{and} \quad 2\zeta_y\Omega_y = C_1 + F_1\,. \tag{5.1}$$

From the equation for x_1,

$$\left(C_1 + \frac{d}{dt}\right)x_1 + J_1x_2 = 0$$

we can interpret C_1^{-1} as a well-known time constant

$$C_1^{-1} = T_y\,. \tag{5.2}$$

Relations involving the symbols C, E, F, J, K can thus be given in a more familiar form if (5.1), (5.2) and their obvious analogues for the z-direction are used. For further simplification we introduce

$$H_y = J_y\Omega_y\,, \qquad H_y' = 2J_y\zeta_y\Omega_y \quad \text{and} \quad H_y'' = J_yT_y^{-1}, \tag{5.3}$$

the similar quantities for the z-direction and adopt the convention that indices are dropped to denote the geometrical mean

$$H^2 = H_yH_z \quad \text{and so on.} \tag{5.4}$$

Since our primary interest is in cases, where J_x is much smaller than J_y and J_z, we will make the approximation

$$K_1 = K_2 = 1. \tag{5.5}$$

5.2. The inequality (3.3) transforms into

$$| H_y - H_z | < [H_y''(H_y' - H_y'')]^{1/2} + [H_z''(H_z' - H_z'')]^{1/2} \sim 2[H''(H' - H'')]^{1/2}. \tag{5.6}$$

The approximation is obtained if we replace $a + b$ by $2(ab)^{\frac{1}{2}}$ and assume that $H_y'' - H_z''$ is a small quantity, a and b being the square roots in the exact inequality.

From (3.5) one obtains

$$| H_y - H_z | < (H_y' + H_z'' - H_y'')^{1/2}(H_z' + H_y'' - H_z'')^{1/2} \sim H \tag{5.7}$$

a slight generalization of the results in [2] to arbitrary moments of inertia, J_y and J_z.

Let R be the ratio of the approximate right member of (5.7) to that of (5.6) and $r = \zeta\Omega T$. Then

$$R = r(2r - 1)^{1/2}. \tag{5.8}$$

The following table shows that R is quite close to 1 for many cases of practical interest:

$r = 0.625$	0.75	1	2.5	5	8.5
$R = 1.25$	1.06	1	1.25	1.66	2.125

The deviation in natural frequency, which is allowed for a stable motion with arbitrary rolling velocities, can certainly not be larger than the allowed deviation in the special case of steady roll. We conclude that Theorem 1, though only sufficient, is rather sharp. One can also infer that in most cases it is sufficient to study only steady roll.

5.3. The results of Sec. 4 will also be reformulated with the aid of 5.1. The numbers b_0 and b_1 from 3.4, which appear in (4.1), are given as

$$b_0 = \Omega_y^2\Omega_z^2 = \Omega^4 \quad \text{and} \quad -b_1 = 2\Omega^2 + (\Omega_y - \Omega_z)^2 - 4\zeta^2\Omega^2 + (T_y^{-1} - T_z^{-1})^2 - 2(T_y^{-1} - T_z^{-1})(\zeta_y\Omega_y - \zeta_z\Omega_z).$$

We will assume for simplicity that

$$J_y = J_z \tag{5.9}$$

for the remaining part of this section, with some obvious exceptions. The equation

$$-b_1 = 2(1 + r^2)\Omega^2 \tag{5.10}$$

defines a number r, which is real if and only if (3.5) is violated, see (5.7). The numbers Ω_1 and Ω_2 defined in (4.1) are

$$\Omega_1 = [1 + r^2 + (2r^2 + r^4)^{1/2}]^{-1/2} \quad \text{and} \quad \Omega_2 = (\cdots)^{1/2}, \tag{5.11}$$

where $(\cdots)$ is repeated from Ω_1. In particular it holds that $\Omega_1 = \Omega_2 = \Omega$, the geometrical mean of Ω_y and Ω_z, when $r = 0$.

Under the assumption that J_iE_i is much larger than C_iF_i, $i = 1, 2$, we can find similar lower bounds for ω_0 from (4.6) or (4.7). For small values of r these bounds are close to Ω_1 given in (5.11). The details are not given since it is easily seen that these simply obtained estimates have a disadvantage for larger values of r. The denominator is the difference between the squares of the natural frequencies, see also (4.10) and (4.11). Better results would be expected if the difference were between unsquared frequencies. A more sophisticated choice of $S_1, \cdots, S_4$ in (4.3) leading to such an estimate is

$$S_1 = J_2K_2^{1/2}\Omega_1, \qquad S_2 = J_1^2J_2K_2^{1/2}\Omega_1^{-1}, \qquad S_3 = J_1K_1^{1/2}\Omega_2 \quad \text{and} \quad S_4 = J_2^2J_1K_1^{1/2}\Omega_2^{-1}.$$

The determinant of (4.3) vanishes when

$$s^4 - 2as^2 + b = 0, \qquad s = \omega_x(H_y - H_z)H^{-1},$$

$$a = 2C_1C_2 + 2F_1F_2 + C_1C_2F_1F_2(\Omega_y\Omega_z)^{-1},$$

$$b = C_1C_2F_1F_2(4\Omega_y^2 - C_1F_1)(4\Omega_z^2 - C_2F_2)(\Omega_y\Omega_z)^{-1}.$$

The result is thus rather complicated. The study of special cases, e.g. when the quantities in (5.3) are all equal, reveals that the consideration of a variable rolling velocity in many cases gives almost the same estimates of allowed rolling velocities as the consideration of steady roll only.

6. Non-constant coefficients in the equations of motion.

6.1. When the coefficients of (1.1) and (1.2) are non-constant, the application of Ljapunov's method offers some theoretical advantages. A linear system $x^{\cdot} + Ax = 0$ may be unstable when A is not constant, even if all the characteristic values of A have positive real parts. Hence Theorem 2 does not hold for non-constant coefficients in (1.2).

This point can be illustrated in connection with a transformation of (1.2). Let the matrix (1.3) be the sum $A_1 + A_2$ of a constant part A_1 and the part A_2 containing the ω_x-terms. The substitution $X = GY$ in (1.2) yields

$$Y^{\cdot} + (G^{-1}G^{\cdot} + G^{-1}A_2G)Y + G^{-1}A_1GY = 0. \tag{6.1}$$

The second term vanishes if the substitution is

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \end{bmatrix}, \qquad \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} a\cos b\delta & c\sin b\delta \\ -a\sin b\delta & c\cos b\delta \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} \tag{6.2}$$

with $\delta = \int \omega_x\,dx$, $a = K_1^{1/2}$, $c = K_2^{1/2}$ and $b = ac$. It is easily seen that (1.2) and (6.1) are both stable or both unstable if they are connected through (6.2). However, A_1 and $G^{-1}A_1G$ have identical characteristic numbers, equal to the characteristic numbers for A when $\omega_x = 0$. These numbers are positive, but (6.1) cannot always be stable according to Theorem 2.

6.2. Let all the coefficients, except K_1 and K_2, of (1.2) be allowed to vary with t and X. We can still use (2.2) and (2.3) and the arithmetic in 3.1. However, there is one point to consider. The forms (2.2) and (2.3) are connected through (2.1) only if q is a constant. The condition (3.3) gurantees that the intervals (q_1, q_2) and (q_3, q_4) have common points, but not that a constant q is among these points. After checking that (3.3) holds, it remains to consider the question just mentioned. One approach is outlined in the next paragraph.

6.3. When K_1, K_2 and/or q are non-constant differentiable functions, the form

$$-x_2^2\frac{d}{dt}(qK_2) - x_4^2\frac{d}{dt}(qK_1)$$

should be added to the form (2.3) to make (2.1), (2.2) and (2.3) compatible. The correction is equivalent to a change of F_1 and F_2 and in order to find sufficient conditions (3.3), the derivatives of qK_1 and qK_2 can be replaced by lower or upper bounds.

6.4. The details of a calculation following the outlined methods must be varied according to the particular case, and it does not seem wise to look for general criteria.

7. Incorporation of some other effects.

7.1. The matrix A defined in (1.3) has a diagonal of zeros. In the deduction of (1.1) we have neglected some aerodynamic effects under the assumption that they are small. If these effects enter into the zero-positions of A, they may be of interest even if they are small. Among the forces and moments given in Ref. [1], the Magnus pitching and yawing moments and the moments due to rolling combined with pitching and yawing angular velocities are the only effects of this type. When they are considered, a matrix

$$B = \omega_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \end{bmatrix} \tag{7.1}$$

should be added to A. The sign of the numbers ϵ_1, ϵ_2 cannot be determined theoretically. The moments are small in the sense that the $| \epsilon_i |$ are much smaller than the other coefficients.

There is no way to form a positive definite Q_1 with constant coefficients, such that Q_2 is also positive definite for all ω_x, when B is added to A. We shall therefore use (2.5) to find quantitative expressions for the obvious statement: The effects can be of importance only if the rolling velocity is high or the stability is poor without the new effects.

7.2. Let Q_1 be defined by (2.2) and Q_2 by (2.1). Let Q_{22} be the part of Q_2 which corresponds to B in (7.1). The other part Q_{21} is the form (2.3). The number s'' in (2.5) is $-\epsilon \mid \omega_x \mid$ if ϵ denotes the largest number of $qK_2 \mid \epsilon_1 \mid$ and $qK_1 \mid \epsilon_2 \mid$. We assume that (3.3) is fulfilled under the assumptions made in Sec. 3. A value of q and a corresponding positive number p can then be chosen such that the characteristic numbers of Q_{21} are all larger than p. From (2.5) we find, using $s' = p$, that our system is stable if

$$\epsilon \mid \omega_x \mid < p. \tag{7.2}$$

The inequality (7.2) relates the allowed rolling velocity to the degree of stability and the size of the new effects introduced.

8. The effect of control systems.

8.1. The interpretations in terms of the natural frequencies and damping factors in Sec. 5 give rise to the following question. What are the relevant quantities if frequency and damping are changed by a controlling device?

Merely to illustrate the possibilities of extensions to such problems, a simple situation will be discussed in this section.

8.2. Let δ_1 and δ_2 be the positions of control surfaces, which are assumed to have no influence on the equations of forces. The equations for $x_1^{\cdot}$ and $x_3^{\cdot}$ in (1.2) are thus not changed. The effect of the control system is to add $H_1\delta_1$ to $x_2^{\cdot}$ and $H_2\delta_2$ to $x_4^{\cdot}$, H_i being positive (constants). We also assume that δ_1 and δ_2 are connected with x_2 and x_4 by linear equations with constant positive coefficients.

$$\left(1 + a_1 \frac{d}{dt}\right)\delta_1 = -\left(b_1 + d_1 \frac{d}{dt}\right)x_2 \quad \text{and} \quad \left(1 + a_2 \frac{d}{dt}\right)\delta_2 = -\left(b_2 + d_2 \frac{d}{dt}\right)x_4$$

representing a simple controlling device.

Define $x_5 = \delta_1 + a_1^{-1} d_1 x_2$ and $x_6 = \delta_2 + a_2^{-1} d_2 x_4$ and let q, q_1 and q_2 be positive numbers. The form

$$Q_1 = x_1^2 + x_3^2 + qK_2x_2^2 + qK_1x_4^2 + q_1x_5^2 + q_2x_6^2$$

yields through (2.1) a form Q_2 which is independent of ω_x . The secular equation for Q_2 splits up into two third degree equations, one of which is

$$\begin{vmatrix} 2C_1 - s & J_1 - qK_2E_1 & 0 \\ J_1 - qK_2E_1 & 2qK_2F_1^* - s & G_1 - qK_2H_1 \\ 0 & G_1 - qK_2H_1 & 2a_1^{-1} - s \end{vmatrix} = 0, \tag{8.1}$$

where $G_1 = q_1a_1^{-2}(a_1b_1 - d_1)$ and $F_1^* = F_1 + H_1a_1^{-1} d_1$.

The other equation takes the same form.

It is found that all the solutions s of (8.1) are positive if and only if

$$\{4C_1qK_2F_1^* - (J_1 - qK_2E_1)^2\}q_1 - a_1C_1(G_1 - qK_2H_1)^2 > 0. \tag{8.2}$$

It holds that when G_1 equals zero ω_y is fed back to δ_1 without filtering, when G_1 is positive the feedback is through a lag filter and when G_1 is negative it is through a lead filter. Only the first two possibilities are considered. If $G_i = 0$ the change of F_i into F_i^* corresponds to the change of the natural frequencies and damping factors by the controlling device. When G_1 is positive the interpretation is somewhat more complicated and will not be discussed here.

We return to (8.2) and observe that it is necessary to have $\{\cdots\} > 0$. If the first term is positive and G_1 non-negative there is, on the other hand, a positive value for q_1 such that $q_1^{-1}a_1C_1(G_1 - qK_2H_1)^2$ equals zero or is as small as we please and hence (8.2) is fulfilled.

The conditions that $4C_1qK_2F_1^* < (J_1 - qK_2E_1)^2$ and $4C_2qK_1F_2^* < (J_2 - qK_1E_2)^2$ are thus both necessary and sufficient for Q_2 to be positive definite for suitable values of q_1 and q_2 . If only the F_i are replaced by F_i^* , $i = 1, 2$, we can then use the deductions in 3.1. In particular (3.3) and Theorem 1 hold with this change of F_1 and F_2 .

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