AN APPROXIMATE SOLUTION FOR THE AXISYMMETRIC JET OF A LAMINAR COMPRESSIBLE FLUID*

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An extension of the modified-Oseen method of Carrier, based on the linearization of the viscous term of the von Mises transformation, is presented. The method is employed to determine the velocity field associated with the laminar axisymmetric jet flow of a compressible gas with an arbitrary but constant external flow. The approximate solution is shown to be in good agreement with the exact numerical calculation of Pai.

1. Introduction. In many boundary layer problems it is not possible to make the assumption of flow similarity. The solution in these cases can be obtained either by laborious finite difference techniques or by resort to approximate solutions. Carrier and Lewis [1], and more recently Carrier [2], have suggested a method of obtaining approximate solutions to problems involving convection and diffusion. This method, termed by Carrier "the modified-Oseen method", overcomes an essential difficulty of integral methods, namely, the generation of reasonable profiles. It is well known that the integral method gives accurate results only if the analytical profiles represent closely the true profiles. According to the modified-Oseen method the convective operator in the original partial differential equation is replaced by a linear one. The resulting equation for the boundary layer problem is the heat conduction equation which can be treated by well-known techniques.

It is the purpose of this paper to indicate a modification of this procedure and to demonstrate its simplicity and accuracy by treating the axisymmetric laminar flow of a compressible gas with arbitrary but constant external flow. The modification is based on the use of the von Mises transformation with a subsequent linearization of the viscous term, rather than the linearization of the convective term. Pai's problem [3], originally treated by a finite difference technique, is considered to illustrate the effectiveness of this method.

2. Analysis. The problem of an axisymmetric jet in a viscous compressible fluid of uniform pressure is described by a set of equations known as the boundary layer equations. This set of equations, expressing the conservation of mass, momentum, and energy, may be written as

$$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial r}(\rho v r) = 0, \tag{1a}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right), \tag{1b}$$

$$\rho u c_p \frac{\partial T}{\partial x} + \rho v c_p \frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right) + \mu\left(\frac{\partial u}{\partial r}\right)^2, \tag{1c}$$

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where x and r represent the axial and radial coordinates, u and v are the axial and radial velocity components, ρ and T are the density and the temperature, and c_p, λ, and μ are the specific heat at constant pressure, and the coefficients of heat conduction and viscosity, respectively.

A dimensionless stream function $\psi = (\Psi/2)^2$ is introduced to obtain the relation

$$\frac{\partial}{\partial r^*}\left(\frac{\Psi}{2}\right)^2 = \rho^* u^* r^*; \qquad \frac{\partial}{\partial x^*}\left(\frac{\Psi}{2}\right)^2 = -\rho^* v^* r^*, \tag{2}$$

where starred quantities defined by $x^* = x/r_i$, $r^* = r/r_i$, $u^* = u/u_0$, $v^* = v/u_0$, $\rho^* = \rho/\rho_0$, and $\mu^* = \mu/\mu_0$ are dimensionless variables, with r_i, u_0, ρ_0, and μ_0 as reference distance, velocity, density, and viscosity coefficient, respectively.

Under the von Mises transformation, with Ψ and x^* as new independent variables, the differential operators become

$$\frac{\partial}{\partial x^*} = \frac{\partial}{\partial x^*} - \frac{2}{\Psi}\rho^* v^* r^* \frac{\partial}{\partial \Psi}; \qquad \frac{\partial}{\partial r^*} = \frac{2}{\Psi}\rho^* u^* r^* \frac{\partial}{\partial \Psi}.$$

In terms of these operators Eq. (1b) reduces to

$$\frac{\partial u^*}{\partial x'} = \frac{4}{\Psi}\frac{\partial}{\partial \Psi}\left[\mu^* \rho^* u^* r^{*2} \frac{1}{\Psi}\frac{\partial u^*}{\partial \Psi}\right], \tag{3}$$

where

$$x' = \left[\frac{\rho_0 u_0 r_i}{\mu_0}\right]^{-1} x^*.$$

The radial coordinate at any given station x' is related to the stream function Ψ by

$$r^{*2} = \int_0^{\Psi} \frac{\Psi\, d\Psi}{\rho^* u^*}$$

or by

$$r^{*2} = \frac{1}{\rho^* u^*}\frac{\Psi^2}{2},$$

wherever $\rho^* u^*$ is not a function of Ψ. The condition of symmetry at the axis, namely $\partial(\rho u)/\partial\psi = 0$ is a particular case where this condition is satisfied identically. An expansion of the term $\rho^* u^* r^{*2}$ in the neighborhood of the axis will, therefore, produce as its leading member the term $\Psi^2/2$. Consider, in the spirit of Carrier, the approximation

$$\mu^* \rho^* u^* r^{*2} = f(x')\left(\frac{\Psi}{2}\right)^2, \tag{4}$$

and let

$$\xi = \int_0^{x'} f(x')\, dx'. \tag{5}$$

Application of Eq. (4) and Eq. (5) to Eq. (3) results in the linear differential equation

$$\frac{\partial u^*}{\partial \xi} = \frac{1}{\Psi}\frac{\partial}{\partial \Psi}\left[\Psi \frac{\partial u^*}{\partial \Psi}\right], \tag{6}$$

which is in the form of the well-known heat conduction equation in cylindrical coordinates. The appropriate boundary conditions are

$$u^*(\Psi, 0) = g(\Psi), \tag{7a}$$

$$u^*(\infty, \xi) = U_\infty^*, \tag{7b}$$

and the regularity condition on the axis, namely,

$$u^*(0, \xi) \neq \infty. \tag{7c}$$

It is clear that Eqs. (6) and (7) yield a solution independent of the energy equation. According to Carslaw and Jaeger [4], the solution for this initial value problem is given by

$$u^*(\Psi, \xi) = \frac{1}{2\xi} \exp\left(-\frac{\Psi^2}{4\xi}\right) \int_0^\infty \exp\left(-\frac{\Psi'^2}{4\xi}\right) I_0\left(\frac{\Psi\Psi'}{2\xi}\right) g(\Psi')\Psi'\, d\Psi' \tag{8}$$

for any specified initial velocity distribution $g(\Psi)$.

The inversion back to the physical coordinates is now indicated. From the definition of the stream function Ψ, the radial coordinate r^* is given by

$$r^{*2} = \int_0^\Psi \frac{\Psi\, d\Psi}{\rho^* u^*}. \tag{9}$$

The variable ρ^* can be obtained as a function of u^* by assuming the Crocco relation. For simple systems this relation gives

$$T = A + Bu + Cu^2$$

subject to the condition of a uniform pressure

$$\rho = [A' + B'u + C'u^2]^{-1}, \tag{10}$$

where A', B', and C' are constants depending on the initial conditions. Once this relation is derived, integration of Eq. (9) can be carried out, since the variable u^* is a known function of Ψ[see Eq. (8)].

The inversion of ξ to x requires the determination of the arbitrary function $f(x')$. Let $E(\Psi, x')$, defined by

$$E = \mu^* \rho^* u^* r^{*2} - f(x')\left(\frac{\Psi}{2}\right)^2 \tag{11}$$

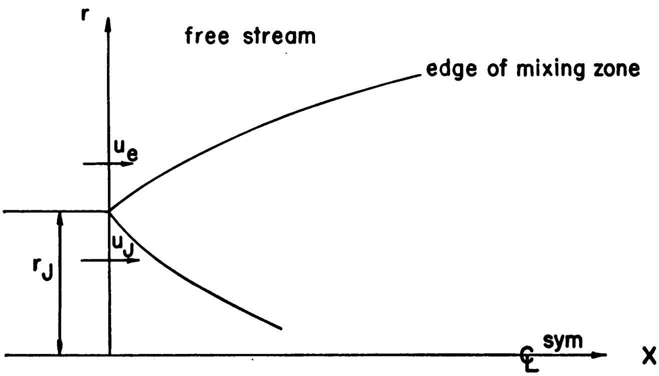

Fig. 1. Schematic diagram of the flow showing the coordinate system.

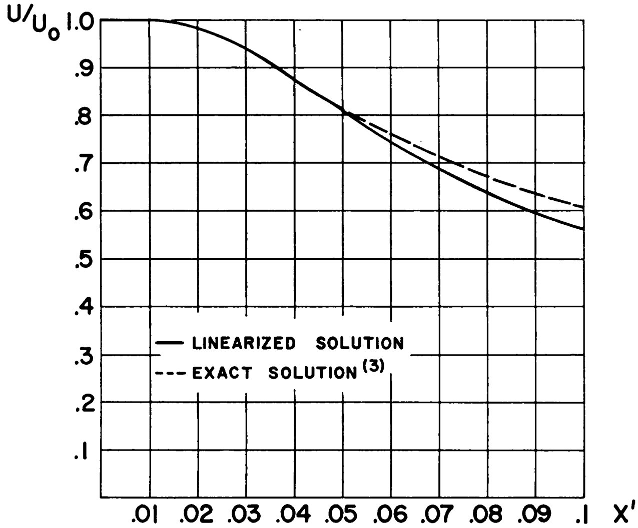

FIG. 2. Velocity distribution along the axis.

be the error involved in this approximation. It is clear from Eq. (11) that for any given value of Ψ, $f(x')$ can be chosen in such a way as to make E vanish. In particular, in the neighborhood of the axis, it can be easily verified from Eq. (9) that

$$f(x') = 2\mu^*(x').$$

While considering a flow field, $f(x')$ is determined in such a way as to yield $E = E_{\min}$ over the region of interest. This requires that E satisfy the relation

$$\int_0^{\Psi_{fs}} E \, d\Psi = 0 \tag{12a}$$

or in terms of the stream function Ψ

$$\int_0^{\Psi_{fs}} E\Psi \, d\Psi = 0, \tag{12b}$$

where the subscript (fs) indicates free stream conditions.

When $f(x')$ is determined, the inversion is completed by the integration

$$x' = \int_0^{\xi} \frac{d\xi}{f(\xi)}. \tag{13}$$

Consideration of Eq. (11) makes it clear that the determination of $f(x')$ is possible only after the inversion of r^* is completed. While this determination can be carried out by a numerical integration, a further approximation is desirable for simplifying the method.

Using Eqs. (11) and (12b), a first approximation for $f(x')$ is given by

$$f(0) = \frac{\int_0^{\Psi_j} (\mu^* \rho^* u^* r^{*2})\Psi \, d\Psi + \int_{\Psi_j}^{\Psi_{fs}} (\mu^* \rho^* u^* r^{*2})\Psi \, d\Psi}{\int_0^{\Psi_{fs}} \left(\frac{\Psi}{2}\right)^2 \Psi \, d\Psi}, \tag{14a}$$

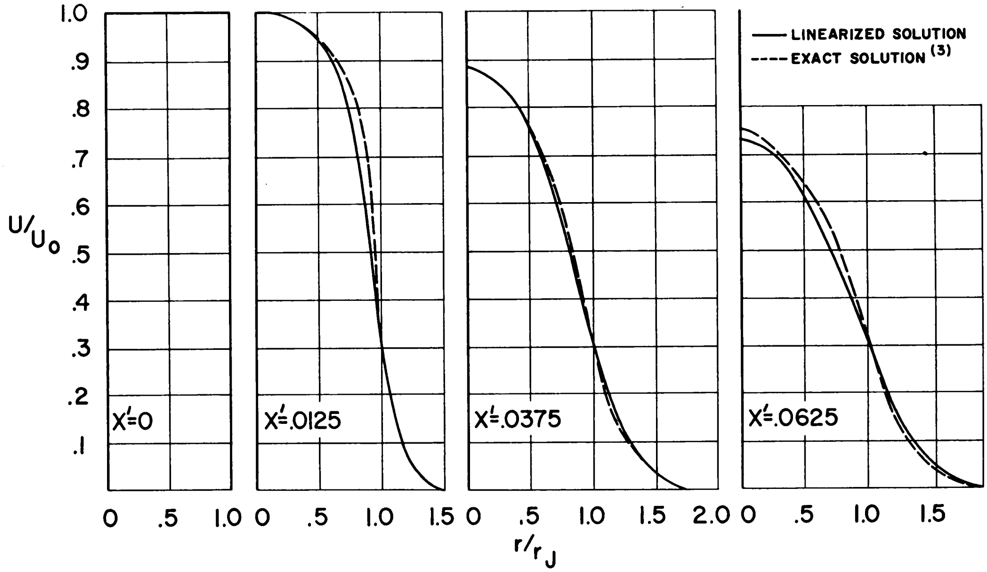

FIG. 3. Velocity distribution for various values of x'.

where allowance is made for possible discontinuity in the quantity $\rho^* u^*$ at the jet boundary $\Psi = \Psi_j$. However, at the injection cross section ($x = 0$), the jet boundary streamline Ψ_j coincides with the free stream value Ψ_{fs}. Therefore, Eq. (14a) reduces to

$$f(0) = \frac{\int_0^{\Psi_j} (\mu^* \rho^* u^* r^{*2}) \Psi \, d\Psi}{\int_0^{\Psi_j} \left(\frac{\Psi}{2}\right)^2 \Psi \, d\Psi}. \tag{14b}$$

The corresponding value of r^* is given by

$$(r^{*2})_{x=0} = \frac{1}{\rho_j^* u_j^*} \frac{\Psi^2}{2}, \tag{15}$$

where $\rho_j^* u_j^*$ is a constant specified by the initial conditions. Combination of Eqs. (14) and (15) gives

$$f(0) = 2\mu_j^*(0)$$

and consequently, from Eq. (13),

$$\xi = 2\mu_j^* x', \tag{16}$$

and the inversion is completed.

3. Numerical example. The application of this method will now be illustrated by considering the injection of a uniform jet into a uniform external flow (Fig. 1). Let u_j^* and u_e^* represent the initial jet and external velocities, respectively. The initial conditions can be taken as

$$g(\Psi)\begin{cases} = u_j^* - u_e^*, & 0 \le \Psi \le \Psi_j, \\ = 0, & \Psi_j < \Psi; \end{cases}$$

subject to these conditions, Eq. (9) reduces to

$$\frac{U}{U_0} = \frac{1}{2\xi} \exp\left(-\frac{\Psi^2}{4\xi}\right) \int_0^{\Psi_j} \exp\left(-\frac{\Psi'^2}{4\xi}\right) I_0\left(\frac{\Psi\Psi'}{2\xi}\right) \Psi' \, d\Psi', \tag{17}$$

where

$$\frac{U}{U_0} = \frac{u^* - u_e^*}{u_j^* - u_e^*}.$$

Eq. (17) is known as the P-function and its values are tabulated elsewhere (Masters, [5]).

The conditions along the axis are subsequently obtained from Eq. (17) by setting $\Psi = 0$. Hence,

$$\frac{U(\xi)}{U_0} = 1 - \exp\left(-\Psi_j^2/(4\xi)\right).$$

To compare this analysis with exact solutions a calculation is carried out for a set of initial conditions which are the same as those of Pai [3], viz.,

$$\left.\begin{array}{l} u^* = 1.1 \\ \rho^* = .5 \end{array}\right\} r^* < 1, \qquad \left.\begin{array}{l} u^* = 1 \\ \rho^* = 1.0 \end{array}\right\} r^* > 1,$$

and

$$\rho^* = [-9.66 + 11.26u^* - .60u^{*2}]^{-1},$$

$$\mu^* = \rho^{*-.76}.$$

Using the approximation given by Eq. (16) the complete solution may be readily obtained with the use of a desk calculator. The results indicated in Figs. 2 and 3 demonstrate a good agreement with the exact solution for a large region of the flow field.

An extension of this method to a non-uniform initial profile can be carried out by expressing the profile as a linear combination of P-functions.

4. Conclusions. The linearization of the equation of motion, based on the von Mises transformation, enables a determination of the velocity distribution in the x, Ψ coordinates independently of the energy equation. The results obtained by utilizing the suggested method indicate a close agreement with the exact numerical calculation of Pai.

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