BUCKLED STATES OF CIRCULAR PLATES*

BY

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1. Introduction. We consider a thin elastic plate of circular shape and constant thickness in static equilibrium under a constant compressive thrust applied at its edge. One possible equilibrium configuration of the plate is a state of uniform compression in which the plate remains plane and contracts radially. It has been observed that this state is unstable when the edge thrust is sufficiently large and then other states of equilibrium occur. They are called buckled states since in them the plate is buckled—i.e. deflected out of its plane. These states occur in pairs in which the deflections are in opposite directions. We shall prove that for every positive integer n a pair of rotationally symmetric buckled states with $n - 1$ internal nodes exist when the thrust is slightly larger than the nth critical value (i.e. eigenvalue) determined by the linear buckling theory. Our analysis is based upon the non-linear von Karman equations of plate theory [1]. We treat both the case in which the edge is clamped and that in which it is simply supported.

The rotationally symmetric buckling of a simply supported plate was previously investigated by K. O. Friedrichs and J. J. Stoker [2]. They showed that the unbuckled state is the only equilibrium configuration when the thrust is less than or equal to the first eigenvalue of the linear buckling theory. They also showed that there is a pair of buckled states with no internal node for all greater values of the thrust and that no other states exist when the thrust is less than or equal to the second eigenvalue. It is reasonable to conjecture, on the basis of experience with the elastica and other problems, that in addition to the unbuckled state there are n pairs of rotationally symmetric buckled states when the thrust is greater than the nth and less than or equal to the $(n + 1)st$ eigenvalue of the linear problem. One pair has no internal nodes, another pair has one internal node and so on up to $n - 1$ internal nodes. The results of Friedrichs and Stoker prove part of this conjecture and our result proves another part, but the conjecture is still not completely proved. A similar conjecture has been proved for certain other non-linear problems by I. I. Kolodner [3] and G. H. Pimbley [4].

Our method of analysis is essentially the bifurcation theory devised by Poincaré to prove the existence of periodic solutions of non-linear initial value problems. This method can also be applied to non-linear boundary value problems. The present paper furnishes two such applications. It was previously employed by J. B. Keller [5] to prove the existence of buckled states of a non-uniform column.

When the Poincaré theory is applicable to a non-linear boundary value problem it can frequently be extended to justify a perturbation solution. Thus in the present paper we justify, in a simple manner, perturbation expansions about each of the eigenvalues of the linear theory. Previously Friedrichs and Stoker [2] employed the rather complicated Schmidt bifurcation theory for this purpose.

*Received July 17, 1961. The research reported in this paper was sponsored in part by the Office of Naval Research under Contract No. Nonr 285(46) and by a grant from North Army Ordnance.

2. The boundary value problem. Let the plate have radius R, thickness h, Youngs' modulus E, Poisson ratio ν and be subjected to a uniform compressive edge thrust $f \geq 0$. Assuming rotationally symmetric deformations, we denote by $w(r)$ the normal displacement of the unstrained midplane and by $\Phi(r)$ the Airy stress function. We then introduce the dimensionless quantities x, λ^2, $U(x)$ and $V(x)$ defined by

$$x = \frac{r}{R} \qquad \lambda^2 = \frac{12(1-\nu^2)}{E}\frac{R^2}{h^2}f,$$

$$U(x) = -[3(1-\nu^2)]^{1/2}\frac{R}{h}\frac{dw(r)}{dr}, \tag{2.0}$$

$$V(x) = -\frac{12(1-\nu^2)}{ER}\frac{R^2}{h^2}\left[\frac{d\Phi(r)}{dr} - fr\right].$$

Then the von Karman equations [1] can be reduced to the form

$$LU(x) + \lambda^2 xU(x) = -V(x)U(x), \tag{2.1}$$

$$LV(x) = U^2(x).$$

Here the differential operator L is defined as

$$L\cdot = x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}x.$$

In the integration which led to (2.1) an arbitrary constant was eliminated by means of the assumed regularity and symmetry of the stresses and displacements at the center of the plate. In addition the symmetry implies

$$U(0) = 0, \qquad V(0) = 0. \tag{2.2a}$$

If the edge of the plate is clamped we have

$$U(1) = 0, \qquad V(1) = 0. \tag{2.2b}$$

The second condition follows from the specification of the radial membrane stress at the edge. The simply supported edge is treated in Appendix II.

Equations (2.1) and (2.2) form a boundary value problem for the determination of the rotationally symmetric equilibrium states of the clamped plate. One solution of this problem, valid for all λ, is the trivial one, $U(x) \equiv V(x) \equiv 0$. This corresponds to a state of pure radial compression of the plate which we call the unbuckled state. If $U(x)$, $V(x)$ is any *other* solution of (2.1) and (2.2) for some λ (i.e. a buckled state) then it clearly follows that $-U(x)$, $V(x)$ is also a solution for the same value of λ. Thus as we stated in the Introduction, the buckled states occur in pairs which differ only in the sign of the normal deflection.

3. Existence of buckled states. The existence of buckled states will now be demonstrated by means of a classical procedure due to Poincaré. For this purpose we consider the initial value problem:

$$\begin{aligned} &\text{(a)} \quad Lu(x) + \lambda^2 xu(x) = -v(x)u(x); \quad u(0) = 0, \quad u'(0) = 1; \\ &\text{(b)} \quad Lv(x) = \xi^2u^2(x) \quad ; \quad v(0) = 0, \quad v'(0) = \eta. \end{aligned} \tag{3.0}$$

Here ξ and η are arbitrary real parameters. Where no confusion can arise we indicate a solution of (3.0) by

$$u(x) \equiv u(\lambda, \xi, \eta; x), \qquad v(x) \equiv v(\lambda, \xi, \eta; x).$$

In Appendix I we prove the following basic theorem relating to such solutions.

Theorem I. For all finite triples (λ, ξ, η) *a unique solution of* (3.0) *exists in* $0 \leqq x \leqq 1$. *The solution is analytic in each of the parameters* λ, ξ, *and* η.

We now seek values of the parameters (λ, ξ, η) such that the corresponding solution of (3.0) satisfies

$$\begin{aligned} u(\lambda, \xi, \eta; 1) &= 0, \\ v(\lambda, \xi, \eta; 1) &= 0. \end{aligned} \tag{3.1}$$

It then follows that a pair of solutions of the boundary value problem (2.1) and (2.2) is given by $U(x) = \pm \xi u(x)$, $V(x) = v(x)$. To solve (3.1) let us choose the special parameter values

$$\xi = \eta = 0.$$

Then (3.0b) is easily integrated and yields $v(x) \equiv 0$. Now (3.0a) reduces to the linear problem

$$Lu(x) + \lambda^2 x u(x) = 0; \qquad u(0) = 0, \qquad u'(0) = 1.$$

The unique solution of this initial value problem is

$$u(x) = \begin{cases} x, & \lambda = 0, \\ J_1(\lambda x), & \lambda \neq 0. \end{cases}$$

Hence $u(1) = 0$ if and only if

$$\lambda = \lambda_n \equiv j_{1,n}, \qquad n = 1, 2, \cdots \tag{3.2}$$

where $j_{1,n}$ is the nth zero of the Bessel function J_1. Of course $v(1) = 0$ for all λ. Thus solutions of (3.0) with $(\lambda, \xi, \eta) = (\lambda_n, 0, 0)$, which satisfy (3.1) are given by

$$\left.\begin{aligned} u^{(n)}(x) &= u(\lambda_n, 0, 0; x) = J_1(\lambda_n x) \\ v^{(n)}(x) &= v(\lambda_n, 0, 0; x) = 0 \end{aligned}\right\} \quad n = 1, 2, \cdots . \tag{3.3}$$

However the corresponding solutions of the boundary value problem (2.1) and (2.2) all reduce to the trivial (unbuckled) solution: $U(x) = \pm\xi u(x) = 0$, $V(x) = v(x) = 0$. The functions $u^{(n)}(x)$ are just the eigenfunctions of the linear buckling theory and the λ_n^2 are the corresponding buckling loads.

We have just shown that the equations (3.1) have a denumerable number of roots $(\lambda_n, 0, 0)$, $n = 1, 2, \cdots$. We shall now try to find other roots in the neighborhoods of these by using the implicit function theorem. To this end we shall evaluate the Jacobian

$$J_\xi = \frac{\partial(u, v)}{\partial(\lambda, \eta)}$$

at $(\lambda, \xi, \eta; x) = (\lambda_n, 0, 0; 1)$. To find the partial derivatives occurring in this Jacobian, we note that by Theorem I the solutions of (3.0) are analytic in (λ, ξ, η). Hence the variational equations satisfied by the partial derivatives (u_λ, v_λ) and (u_η, v_η) can be obtained by formal differentiation of the equations and initial conditions in (3.0). Then with the notation

$$\frac{\partial}{\partial\lambda} u(\lambda, \xi, \eta; x) \Bigg|_{\substack{\lambda=\lambda_n \\ \xi=0 \\ \eta=0}} = u_\lambda^{(n)}(x), \quad \text{etc.},$$

we obtain from (3.3) and (3.0)

$$\begin{aligned} &\text{(a)} \quad Lu_\lambda(x) + \lambda_n^2 x u_\lambda(x) = -[2\lambda_n x + v_\lambda(x)]J_1(\lambda_n x), \qquad u_\lambda(0) = u_\lambda'(0) = 0; \\ &\text{(b)} \quad Lv_\lambda(x) = 0, \qquad v_\lambda(0) = v_\lambda'(0) = 0; \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\text{(a)} \quad Lu_\eta(x) + \lambda_n^2 x u_\eta(x) = -v_\eta(x) J_1(\lambda_n x), \qquad u_\eta(0) = u_\eta'(0) = 0; \\ &\text{(b)} \quad Lv_\eta(x) = 0, \qquad v_\eta(0) = 0, \quad v_\eta'(0) = 1. \end{aligned} \tag{3.5}$$

Here we have dropped the superscript for convenience and each system holds for $n = 1, 2, \cdots$.

From (3.4b) we find that $v_\lambda(x) \equiv 0$ and hence the Jacobian J_ξ, when evaluated at each root $(\lambda_n, 0, 0)$, reduces to

$$J_\xi \equiv \frac{\partial(u, v)}{\partial(\lambda, \eta)} = u_\lambda(1) v_\eta(1).$$

From (3.5b) it follows that $v_\eta(x) = x$ and thus $J_\xi = u_\lambda(1)$. If $u_\lambda(1) = 0$ we can conclude from (3.4) that $u_\lambda(x)$ is a solution of the boundary value problem

$$Lu_\lambda(x) + \lambda_n^2 x u_\lambda(x) = -2\lambda_n x J_1(\lambda_n x), \qquad u_\lambda(0) = u_\lambda(1) = 0.$$

However for such a solution to exist the inhomogeneous term must be orthogonal to all non-trivial solutions of the homogeneous adjoint problem. These solutions are $J_1(\lambda_n x)$ and so the orthogonality condition requires the vanishing of

$$2\lambda_n \int_0^1 x J_1^2(\lambda_n x)\, dx.$$

Since this is impossible, it follows that $u_\lambda(1) \neq 0$ and hence $J_\xi \neq 0$.

Thus the implicit function theorem is applicable and implies that the system (3.1) can be solved uniquely in a sufficiently small neighborhood of each root λ_n, 0, 0. The solutions are *unique* functions

$$\lambda = \lambda_n(\xi), \qquad \eta = \eta_n(\xi), \qquad |\xi| < \epsilon_n; \qquad n = 1, 2, \cdots, \tag{3.6}$$

which satisfy $\lambda_n(0) = \lambda_n$, $\eta_n(0) = 0$, $u(\lambda_n(\xi), \xi, \eta_n(\xi); 1) = 0$ and $v(\lambda_n(\xi), \xi, \eta_n(\xi); 1) = 0$. Furthermore, by the analyticity of u and v in (λ, ξ, η), it follows that these functions are analytic in ξ for $|\xi| < \epsilon_n$. In addition it follows from (3.0) that

$$\begin{aligned} u(\lambda, \xi, \eta; x) &= u(\lambda, -\xi, \eta; x), \\ v(\lambda, \xi, \eta\, x) &= v(\lambda, -\xi, \eta; x). \end{aligned} \tag{3.7}$$

From (3.7) and the uniqueness of $\lambda_n(\xi)$ and $\eta_n(\xi)$, it follows that

$$\lambda_n(-\xi) = \lambda_n(\xi) \quad \text{and} \quad \eta_n(-\xi) = \eta_n(\xi).$$

We may summarize our results as follows.

Theorem II. For each positive integer n there exists an $\epsilon_n > 0$ *and two even analytic functions* $\lambda_n(\xi)$ *and* $\eta_n(\xi)$ *defined for* $|\xi| < \epsilon_n$ *and satisfying*

$$\lambda_n(0) = \lambda_n \equiv j_{1,n}, \qquad \eta_n(0) = 0. \tag{3.8}$$

A one parameter pair of solutions of (2.1) *and* (2.2) *exists for* $|\xi| < \epsilon_n$. *They are given by*

$$\begin{aligned} U(x) &= \pm\xi u[\lambda_n(\xi), \xi, \eta_n(\xi); x] \equiv \pm\xi u^{(n)}(\xi, x), \\ V(x) &= \quad v[\lambda_n(\xi), \xi, \eta_n(\xi); x] \equiv \quad v^{(n)}(\xi, x). \end{aligned} \tag{3.9}$$

4. Properties of the buckled states. Since $\lambda_n(\xi)$ and $\eta_n(\xi)$ are analytic even functions of ξ satisfying (3.8) they have expansions of the form

$$\begin{aligned} \lambda_n(\xi) &= \lambda_n + \sum_{k=1}^{\infty} \lambda_{n,k}\xi^{2k}, \\ & \qquad\qquad\qquad n = 1, 2, \cdots \\ \eta_n(\xi) &= 0 + \sum_{k=1}^{\infty} \eta_{n,k}\xi^{2k}. \end{aligned} \tag{4.0}$$

Moreover, the solutions of (3.0) are analytic in the parameters (λ, ξ, η) and even in ξ. Therefore with the aid of (4.0) the solutions (3.9) have the expansions

$$\begin{aligned} u^{(n)}(\xi; x) &= J_1(\lambda_n x) + \sum_{k=1}^{\infty} u_k^{(n)}(x)\xi^{2k}, \\ & \qquad\qquad\qquad n = 1, 2, \cdots \\ v^{(n)}(\xi; x) &= 0 + \sum_{k=1}^{\infty} v_k^{(n)}(x)\xi^{2k}. \end{aligned} \tag{4.1}$$

The series in (4.0) and (4.1) converge in some interval $|\xi| < \epsilon_n$.

The coefficients in the above expansions can be determined by using (4.0), (4.1) and (3.9) in (2.1) and (2.2). The resulting system for $k = 1$ is:

$$Lu_1^{(n)}(x) + \lambda_n^2 x u_1^{(n)}(x) = -[2\lambda_n\lambda_{n,1}x + v_1^{(n)}(x)]J_1(\lambda_n x),$$
$$u_1^{(n)}(0) = u_1^{(n)}(1) = 0; \tag{4.2a}$$

$$Lv_1^{(n)}(x) = J_1^2(\lambda_n x), \qquad v_1^{(n)}(0) = v_1^{(n)}(1) = 0. \tag{4.2b}$$

The solution of (4.2b) can be written as:

$$v_1^{(n)}(x) = -\int_0^1 g(x, \xi)J_1^2(\lambda_n\xi)\, d\xi, \tag{4.3}$$

where

$$g(x, \xi) = \begin{cases} \dfrac{1}{2}\left(\dfrac{1}{x} - x\right)\xi, & 0 \leqq \xi < x \leqq 1; \\[2ex] \dfrac{1}{2}x\left(\dfrac{1}{\xi} - \xi\right), & 0 \leqq x < \xi \leqq 1. \end{cases} \tag{4.4}$$

Since λ_n^2 is an eigenvalue, (4.2a) has a solution if and only if the appropriate orthogonality condition is satisfied. This condition serves to determine $\lambda_{n,1}$ as

$$\lambda_{n,1} = \frac{\int_0^1 \int_0^1 g(x, \xi) J_1^2(\lambda_n \xi) J_1^2(\lambda_n x) \, d\xi \, dx}{2\lambda_n \int_0^1 x J_1^2(\lambda_n x) \, dx} . \tag{4.5}$$

Here we have used (4.3). Since $g(x, \xi) \geqq 0$ it follows that $\lambda_{n,1} > 0$. Then from (4.0) we may conclude that for some $\epsilon_n' \leqq \epsilon_n$, $\lambda_n(\xi)$ is an increasing function of $|\xi|$ in $0 \leqq |\xi| < \epsilon_n'$. Thus the solutions (3.9) of Theorem II exist for loads $\lambda^2 \geqq \lambda_n^2$, i.e. loads greater than the linear buckling loads. Our existence proof applies only for loads slightly greater than the linear buckling loads, but we conjecture that they actually exist for all greater loads.

By using (4.5) we can determine the solution $u_1^{(n)}(x)$ of (4.2a) to within an arbitrary multiple of $J_1(\lambda_n x)$. The multiplying constant can be determined by considering the system for $k = 2$. In principle this procedure can be continued to determine the coefficients $\lambda_{n,k}$, $u_k^{(n)}(x)$ and $v_k^{(n)}(x)$ for all $k = 1, 2, \cdots$. It is standard perturbation procedure which has been applied by Friedrichs and Stoker [2] to the simply supported plate problem for $n = 1$. In the present case, it is rigorously justified for some interval $\lambda_n \leqq \lambda < \lambda_n + \delta$ by the analyticity properties which have been demonstrated. It was justified in [2] for $n = 1$ by means of the Schmidt bifurcation theory.

Finally we observe from (4.1) that for sufficiently small ϵ_n''

$$u^{(n)}(\xi; x) = J_1(\lambda_n x) + 0(\xi^2), \qquad 0 \leqq |\xi| < \epsilon_n''.$$

Thus the number of zeros of $u^{(n)}(\xi; x)$ in $0 < x < 1$ is the same as that of $J_1(\lambda_n x)$ in this interval. All of the above results may now be summarized as the

Corollary: *For each* $n = 1, 2, \cdots$ *there exists an* ϵ_n'' *satisfying* $0 < \epsilon_n'' \leqq \epsilon_n$ *such that* $\lambda_n(\xi)$ *is an increasing function of* $|\xi|$ *in* $0 < |\xi| < \epsilon_n''$. *For any* λ *satisfying* $\lambda_n \leqq \lambda < \lambda_n(\epsilon_n'')$ *there exists a pair of solutions of* (2.1) *and* (2.2) *such that* $U(x) = \pm \xi u^{(n)}(\xi, x)$ *has* $n - 1$ *simple zeros in* $0 < x < 1$.

Appendix I

Proof of Theorem I. With the introduction of the new variables

$$y(x) = \frac{1}{x}(xu)', \qquad z(x) = \frac{1}{x}(xv)'$$

the initial value problem (3.0) can be replaced by the equivalent first order system

$$\begin{aligned}
&\text{(a)} && u' = -\frac{1}{x} u + y, && u(0) = 0; \\
&\text{(b)} && y' = -\frac{1}{x} uv - \lambda^2 u, && y(0) = 2; \\
&\text{(c)} && v' = -\frac{1}{x} v + z, && v(0) = 0; \\
&\text{(d)} && z' = \frac{\xi^2}{x} u^2, && z(0) = 2\eta.
\end{aligned} \tag{A.0}$$

A formal power series solution of (A.0) is given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^{2k+1}, \qquad y(x) = 2 \sum_{k=0}^{\infty} (k+1) a_k x^{2k};$$
$$v(x) = \sum_{k=0}^{\infty} b_k x^{2k+1}, \qquad z(x) = 2 \sum_{k=0}^{\infty} (k+1) b_k x^{2k}; \tag{A.1}$$

provided that

(a) $$a_0 = 1, \qquad b_0 = \eta; \tag{A.2}$$

and

(b) $$\left.\begin{aligned} a_{k+1} &= \frac{-1}{4(k+1)(k+2)} \left[\lambda^2 a_k + \sum_{j=0}^{k} a_j b_{k-j}\right] \\ b_{k+1} &= \frac{\xi^2}{4(k+1)(k+z)} \sum_{j=0}^{k} a_j a_{k-j} \end{aligned}\right\} k = 0, 1, 2, \cdots . \tag{A.2}$$

The convergence of these series as well as the analyticity properties will follow from

Lemma 1. *Let*

$$m(\eta) \equiv 1 + |\eta|, \qquad M(\lambda, \xi, \eta) \equiv \tfrac{1}{8}[\lambda^2 + m(\eta)(1 + \xi^2)].$$

Then for every finite triple (λ, ξ, η) *the coefficients* $a_k(\lambda, \xi, \eta)$, $b_k(\lambda, \xi, \eta)$ *defined in* (A.2) *satisfy*

$$|a_n| \leqq mM^n, \qquad |b_n| \leqq mM^n; \qquad n = 0, 1, 2, \cdots .$$

Proof. We proceed by induction. From (A.2a) we have $|a_0| = 1 \leqq m$ and $|b_0| = |\eta| < m$ and so the result holds for $n = 0$. Assuming it true for all $n \leqq k$ we obtain from the first equation in (A.2b)

$$\begin{aligned} |a_{k+1}| &\leqq \frac{1}{4(k+1)(k+2)} \left[\lambda^2 mM^k + \sum_{j=0}^{k} m^2 M^k\right], \\ &\leqq \frac{1}{4(k+2)} \left[\frac{\lambda^2}{k+1} + m\right] mM^k, \\ &\leqq \frac{1}{8M} [\lambda^2 + m] mM^{k+1}, \\ &\leqq mM^{k+1}. \end{aligned}$$

Similarly from the second equation in (A.2b)

$$\begin{aligned} |b_{k+1}| &\leqq \frac{\xi^2}{4(k+2)} m^2 M^k, \\ &\leqq \frac{m\xi^2}{8M} mM^{k+1}, \\ &\leqq mM^{k+1}. \end{aligned}$$

The proof of the lemma is now complete.

We introduce

$$R(\lambda, \xi, \eta) \equiv M^{-1/2}(\lambda, \xi, \eta).$$

Then by lemma 1 and comparison with the geometric series $mx \sum_{n=0}^{\infty} (x/R)^{2n}$, it follows that the power series for $u(x)$ and $v(x)$ given in (A.1) converge uniformly and absolutely in every interval

$$| x | \leqq R(\lambda, \xi, \eta) - \delta, \qquad 0 < \delta < R(\lambda, \xi, \eta). \tag{A.3}$$

Thus $u(x)$ and $v(x)$ are analytic in $| x | < R$ and we note that they have odd order zeros at $x = 0$. Hence $y = u' + u/x$ and $z = v' + v/x$ are also analytic in $| x | < R$ and must have convergent power series expansions given in (A.1). The existence of a solution of (A.0) in the interval $| x | < R(\lambda, \xi, \eta)$ is thus established and we note that $R > 0$ for finite (λ, ξ, η).

Let $(\lambda_0, \xi_0, \eta_0)$ be an arbitrary fixed set of finite parameter values and R_0 be a fixed number in $0 < R_0 < R(\lambda_0, \xi_0, \eta_0)$. Then for all x in $| x | \leqq R_0$ it follows from Lemma 1 that:

$$| a_n(\lambda, \xi, \eta)x^{2n} | \leqq m(\eta_0), \qquad | b_n(\lambda, \xi, \eta)x^{2n} | \leqq m(\eta_0); \tag{A.4}$$

for all (λ, ξ, η) in $| \lambda | \leqq | \lambda_0 |, | \xi | \leqq | \xi_0 |, | \eta | \leqq | \eta_0 |$. However a simple induction assures us that $a_k(\lambda, \xi, \eta)$ and $b_k(\lambda, \xi, \eta)$ are both polynomials of degrees not exceeding $2k$ in λ and ξ and not exceeding k in η for $k = 1, 2, \cdots$. The Weierstrass convergence theorem can now be applied to the series in (A.1). It implies that $u(\lambda, \xi, \eta; x)$ and $v(\lambda, \xi, \eta; x)$ are analytic functions of (λ, ξ, η) in the above intervals. Since $(\lambda_0, \xi_0, \eta_0)$ was arbitrary we have shown that the solution (A.1) is in fact entire in (λ, ξ, η).

We shall now show that this solution is unique. Let us assume the existence of two solutions (u_1, y_1, v_1, z_1) and (u_2, y_2, v_2, z_2) in some common interval $0 \leqq x \leqq \epsilon$. Then from the definitions

$$U(x) = u_1(x) - u_2(x), \qquad Y(x) = y_1(x) - y_2(x),$$
$$V(x) = v_1(x) - v_2(x), \qquad Z(x) = z_1(x) - z_2(x),$$

we obtain with the aid of (A.0)

$$\begin{aligned}
&\text{(a)} \quad & U' &= -\frac{1}{x} U + Y & U(0) &= 0;\\
&\text{(b)} \quad & Y' &= -\left(\frac{v_2}{x} + \lambda^2\right)U - \frac{u_1}{x} V, & Y(0) &= 0;\\
&\text{(c)} \quad & V' &= -\frac{1}{x} V + Z, & V(0) &= 0;\\
&\text{(d)} \quad & Z' &= \xi^2\left(\frac{u_1 + u_2}{x}\right)U, & Z(0) &= 0.
\end{aligned} \tag{A.5}$$

Now let

$$W(x) = U^2(x) + V^2(x) \geqq 0,$$

and we find from (A.5a) and (A.5c) that:

$$\frac{1}{2}\frac{d}{dx} W = -\frac{1}{x} W + UY + VZ, \qquad W(0) = 0. \tag{A.6}$$

It follows from (A.0) that $u_1(x)$, $u_2(x)$ and $v_2(x)$ have derivatives in $0 < x \leqq \epsilon$ and right sided derivatives at $x = 0$. Then, recalling that $u_1(0) = u_2(0) = v_2(0) = 0$, there exists some finite constant $k_1 > 0$ such that

$$\max_{0 \leq x \leq \epsilon} \left(\left| \frac{u_1(x)}{x} \right|, \left| \frac{u_2(x)}{x} \right|, \left| \frac{v_2(x)}{x} \right| \right) \leqq k_1 .$$

With the definition

$$k_2 = \max (k_1 + \lambda^2, 2\xi^2 k_1),$$

we obtain by integration of (A.5b) and (A.5d):

$$| Y(x) | \leqq k_2 \int_0^x (| U(x') | + | V(x') |) \, dx',$$

$$| Z(x) | \leqq k_2 \int_0^x | U(x') | \, dx'.$$

Using these bounds in (A.6) implies

$$\frac{1}{2} \frac{d}{dx} W(x) \leqq -\frac{1}{x} W(x) + k_2(| U(x) | + | V(x) |) \int_0^x (| U(x') | + | V(x') |) \, dx'. \qquad \text{(A.7)}$$

Now let ϵ_0 be any positive number such that

$$\epsilon_0^2 \leqq \min \left(\epsilon^2, \frac{1}{2k_2} \right),$$

and let x_0 in $0 < x \leqq \epsilon_0$ be any point for which

$$W(x_0) = \max_{0 \leq x \leq \epsilon_0} W(x).$$

Then by Schwartz's inequality,

$$| U(x) | + | V(x) | \leqq 2^{1/2} W^{1/2}(x) \leqq 2^{1/2} W^{1/2}(x_0), \qquad 0 \leqq x \leqq \epsilon_0 .$$

With $x = x_0$ in (A.7) we now obtain, using the above inequality twice,

$$\frac{1}{2} \frac{d}{dx} W(x_0) \leqq \left[2k_2 x_0 - \frac{1}{x_0} \right] W(x_0). \qquad \text{(A.8)}$$

By the choice of ϵ_0 we must have

$$\left[2k_2 x_0 - \frac{1}{x_0} \right] < 0, \quad \text{for all} \quad x_0 \quad \text{in} \quad 0 \leqq x_0 \leqq \epsilon_0 .$$

Now as $W(x)$ is positive for any x in $0 < x \leqq \epsilon_0$ it must have a positive maximum in this interval. Since this must occur at $x = x_0$ we have from (A.8) that $dW(x_0)/dx < 0$. But this is a contradiction of the fact that at a maximum in $0 < x \leqq \epsilon_0$, $dW/dx \geqq 0$. Thus $W(x) \equiv 0$ in $0 \leqq x \leqq \epsilon_0$ and the uniqueness is established.

We have now proved that the initial value problem (3.0) and the equivalent problem for the first order system (A.0) each possess a unique solution in some finite interval $0 \leqq x \leqq \epsilon_0$, $\epsilon_0 > 0$. This solution is analytic in the parameters λ, ξ and η. We now use the value of the solution at $x = \xi_0$ as initial data to extend the solution to the region $x > \epsilon_0$. Such an extension exists, is unique and analytic in the parameters so long as the right sides of (A.0) are finite. Since $x > 0$, they can cease being finite only if some component of the solution becomes infinite. We shall now show that this does not occur for any $x > 0$ and therefore the solution can be continued indefinitely.

To show that the solution is finite for all finite x, we integrate each of the equations (A.0) from 0 to x and use the initial conditions to obtain

$$\begin{aligned}
z &= 2\eta + \xi^2 \int_0^x x^{-1}u^2\, dx, \\
v &= x^{-1} \int_0^x xz\, dx = \eta\, x + \frac{\xi^2}{2x} \int_0^x (x^2 - s^2)s^{-1}u^2(s)\, ds, \\
y &= 2 - \int_0^x (x^{-1}v + \lambda^2)u\, dx, \\
&= 2 - (\eta + \lambda^2) \int_0^x u\, dx - \frac{\xi^2}{2} \int_0^x t^{-2} \int_0^t (t^2 - s^2)s^{-1}u^2(s)u(t)\, ds\, dt \\
u &= x^{-1} \int_0^x xy\, dx = x - \frac{(\eta + \lambda^2)}{2x} \int_0^x (x^2 - s^2)u(s)\, ds \\
&\qquad - \frac{\xi^2}{4x} \int_0^x (x^2 - t^2)t^{-2} \int_0^t (t^2 - s^2)s^{-1}u^2(s)u(t)\, ds\, dt.
\end{aligned} \tag{A.9}$$

In the last three equations we have substituted from the preceding equations and also interchanged orders of integration to simplify certain repeated integrals. We now rewrite the last equation in the form

$$u(x) = x - \int_0^x k(x, t)u(t)\, dt. \tag{A.10}$$

Here $k(x, t)$ is defined by

$$k(x, t) = \frac{(x^2 - t^2)}{2x} \left[\eta + \lambda^2 + \frac{\xi^2}{2} \int_0^t (1 - t^{-2}s^2)s^{-1}u^2(s)\, ds \right]. \tag{A.11}$$

From (A.10) we wish to conclude that $u(x)$ is finite for every finite value of x. To do so we first suppose that $| k(x, t) |$ is bounded by a constant k_0 for all t and x satisfying $0 \leqq t \leqq x \leqq x_0$. Then (A.10) yields

$$| u(x) | \leqq x_0 + k_0 \int_0^x | u |\, dx, \qquad 0 \leqq x \leqq x_0 . \tag{A.12}$$

From (A.12) it follows that

$$| u(x) | \leqq x_0 e^{k_0 x}, \qquad 0 \leqq x \leqq x_0 . \tag{A.13}$$

Let us now consider the case in which k is not finite for every x. From (A.11) we see that the least value of x for which k can become infinite is the least value of x at which u becomes infinite. Let this value of x be x_0. Then from (A.11) the singular part of k is positive. But then (A.10) shows that $u(x_0)$ is not positively infinite since the singular term on the right side would then be negative. Similarly $u(x_0)$ is not negatively infinite. Thus $u(x_0)$ is finite so no x_0 exists at which u becomes infinite. Since $u(x)$ is finite for all x, it follows from (A.9) that z, v and y are also finite. Therefore the solution can be continued indefinitely.

In conclusion we observe from (A.9) that if $\eta > 0$ then $v(x) > 0$ for all $x > 0$. Therefore (3.1) cannot be satisfied if $\eta > 0$ so solutions of (3.0) with $\eta > 0$ cannot solve the buckling problem (2.1) and (2.2).

Appendix II

The simply supported plate. If the edge of the plate is simply supported then the boundary value problem is given by Eqs. (2.1), (2.2a) and, in place of (2.2b),

$$U'(1) + \nu U(1) = 0, \qquad V(1) = 0. \tag{2.2b_s}$$

The analysis proceeds as in Sec. 3 and 4 but now Eqs. (3.1) are replaced by

$$f(\lambda, \xi, \eta) \equiv u'(\lambda, \xi, \eta; 1) + \nu u(\lambda, \xi, \eta; 1) = 0, \qquad (3.1)_s$$
$$v(\lambda, \xi, \eta; 1) = 0.$$

Then parameter values $(\lambda_n, 0, 0)$ which satisfy $(3.1)_s$ are in place of (3.2),

$$\lambda_n = k_{1,n}, \qquad n = 1, 2, \cdots, \qquad (3.2)_s$$

where $k_{1,n}$ is the nth root of

$$\lambda J_1'(\lambda) + \nu J_1(\lambda) = 0.$$

There is again no root for $\lambda = 0$ since $\nu > -1$. Now solutions of (3.0) which satisfy $(3.1)_s$ are given by (3.3), provided the λ_n are as above. The solutions again correspond to the trivial (unbuckled) solution $U(x) \equiv V(x) \equiv 0$, for the simply supported plate.

We now consider the Jacobian

$$J_\xi \equiv \frac{\partial(f, v)}{\partial(\lambda, \eta)}$$

at $(\lambda, \xi, \eta, x) = (\lambda_n, 0, 0, 1)$. Using (3.4) and (3.5) we find that

$$J_\xi = f(\lambda_n, 0, 0) = u'(1) + \nu u(1).$$

If $J_\xi = 0$ we obtain a contradiction exactly as before. Hence by an application of the implicit function theorem and Theorem I we obtain Theorem II_s : the analogue of Theorem II in which $j_{1,n}$ is replaced by $k_{1,n}$ and in which $U(x)$ and $V(x)$ are a one parameter pair of solutions of (2.1), (2.2a) and $(2.2\text{b})_s$ (i.e. of the simply supported plate problem).

The properties of these solutions are exactly as described in Sec. 4 where λ_n is the value in $(3.2)_s$ and the boundary condition at $x = 1$ in (4.2a) is replaced by

$$\frac{d}{dx} u_1^{(n)}(1) + \nu u_1^{(n)}(1) = 0.$$

In particular $v_1^{(n)}(x)$ is still given by (4.3) and $\lambda_{n,1}$ by (4.5).

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