CAMOUFLAGING ELECTRICAL 1-NETWORKS AS GRAPHS*

BY

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Summary. Lately it has become fashionable to develop the theory of electrical networks by starting first with a detailed, scholarly exposition of the theory of graphs (containing both nodes and branches), as given in textbooks on algebraic topology. However, when the graph analysis finally gets down to the study of actual electrically excited networks, it will invariably be found that all traces of the concepts of nodes and their associated "incidence-matrices" have disappeared from the final $e = zi$ and other equations of state that are to be solved. In their place only the branches and the associated "connection-matrices" have been put to use, all still wrapped up however in an ill-fitting graph terminology. Of course such legerdemain had to be resorted to, since graph theorists happened to pick the wrong topological model for an electrical network. A graph possesses enough structure to propagate an electromagnetic wave and not an electrical current.

This paper points out that the proper stage of an electrically excited e, i, z system is not a graph, nor a planar polyhedron, but a 1-network,—a configuration that contains branches only and no nodes, or planes. The current-law of Kirchhoff prohibits the presence of nodes; while the voltage-law of Kirchhoff prohibits the presence of planes. Hence only a pure branch-network (a 1-network) can automatically satisfy both laws of Kirchhoff, provided the branches are properly organized. Unfortunately texts on algebraic topology never deal systematically with the theory of 1-networks (or of single p-networks), since their interest is not the theory of electricity. Topologists always start out with the theory of the more general graphs, or of polyhedra; hence the confusion of electrical engineers.

Over two decades ago the author, in his book "Tensor Analysis of Networks", developed a branch theory of electrical networks. The latter also were endowed with variable connectedness ("tearing and interconnecting"), in order to allow all possible types of network manipulations and operations. The central backbone of the "orthogonal" theory is a square, nonsingular connection tensor C, and its inverse A. Their four rectangular subdivisions C_0, C_c and A^0, A^c, orthogonal to each other, dictate the existence of four extreme types of switching conditions in any electrical network. It is one or more of these four rectangular matrices that are actually being used today by graph theorists in their final equations, but denoted by different symbols and misnamed by them "incidence matrices", or "reduced" incidence matrices. The latter, however, are radically different topological concepts, that really belong to a graph and never to a 1-network. Hence the correct branch-network finally used (after introducing and then ignoring the nodes), is also misnamed a graph.

Since a 0-network, as well as each higher-dimensional p-network, also possesses a square, nonsingular connection tensor C, it is possible to pass an electromagnetic wave—but not an electric current—across a combination of two or more p-networks, called

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a polyhedron. (Such animated polyhedra can be used for modelling complex non-electrical problems that have a large number of independent variables.) This paper develops the connection-matrices and incidence-matrices associated with a graph, as well as with a planar polyhedron. The paper also shows, with the aid of signal-flow diagrams, the relations of the two types of matrices (or rather tensors) to each other, to the laws of Kirchhoff, and to Stokes' Theorem. Such a treatment unmistakably pin-points the electrical 1-network within its customarily camouflaged polyhedral environment.

Unexcited networks as polyhedra. To avoid any misunderstanding, it should be strongly emphasized that all statements made in this paper assume that the network possesses impedance operators z and is excited by electrical currents (or by electromagnetic waves). The paper does not apply to *unexcited* networks (unless otherwise stated), nor to impedanceless switching circuits. Also the statements do not apply to networks excited by nonelectrical quantities.

That is, before its excitation any electrical network may be looked upon as a polyhedron: namely as a collection of 0-simplexes (nodes), 1-simplexes (branches), 2-simplexes (triangular planes), 3-simplexes (tetrahedra) etc., all interconnected into one structure. It is the argument of the paper that after excitation by conventional voltages and currents, all simplexes (all nodes, planes, etc.) disappear from view and only the branches or 1-simplexes are recognized by the electrical e, i, z system.

The topological problem. The topological problem confronted by the electrical engineer is as follows:

1) Given a set of wires with electrical currents and voltages superimposed. Since all laws of electricity are assumed to be known, the superimposed "algebraic structure" is thereby defined. It deals with the algebra associated with the three electrical concepts e, i and z.

2) Establish *by trial and error* an underlying topological structure, which can automatically adjust itself to all the algebraic peculiarities of the superimposed electrical e, i, z system. There is no *a priori* reason why one of the possible structures should fit the algebra rather than some other.

Three approaches. Graph theorists appropriate, as the final goal of their search, the *very first* topological structure which is being encountered whenever an elementary textbook on algebraic topology (such as Ref. 6) is opened. That structure happens to be the "graph". Irrespective of the subtle nature of the superimposed electricity, graph-theorists force the electric current, by all types of contortions, to fill up the entire available graph-structure, as if the latter were a Procrustean bed.

To escape the numerous contradictions encountered in graph theory, many electrical engineers enlarge the graph with planes (2-simplexes). The resultant planar polyhedron, however, introduces even more contradictions, without removing the existing ones.

The present paper summarizes a painstaking and long search for a suitable underlying topological structure that can adjust itself automatically to all the idiosyncracies of the superimposed electrical current. Since the algebraic structure of electricity flowing in networks is rather analogous to the algebra of *"exterior" differential forms* (employing skew-symmetric tensors of various ranks) discussed in more advanced and more modern texts on differential topology (see Ref. 7), the resultant topological model of an electrical network is also of a more advanced and more sophisticated nature. Instead of being a rigid graph, or a rigid planar polyhedron, the electrical-network model presented is a flexible "orthogonal" 1-network that contains branches (1-simplexes) only, and that can be freely torn and interconnected in all possible ways.

It is disconcerting that no textbook on algebraic topology deals in detail with structures that contain only *one* type of p-simplexes. The reason must be that these texts do not deal with the theory of electricity, and hence with the highly specialized topological structures needed by electricity. Apparently the electrical engineer must do his own specialized research on algebraic topology to suit his own peculiar needs.

Graphs as carriers of masses. In order to compare the 1-network model to be presented with the graph model, a short outline of the latter is given first.

A graph (Fig. 2a) is a structure consisting of oriented, interconnected 0-simplexes (vertices) and 1-simplexes (branches) (Fig. 1a and 1b). It is possible to put arbitrary *weights* upon each branch by assigning a different number to each branch. It is also possible to put arbitrary weights upon each node by assigning a different number to each node. For instance, three of the node-weights may be zero and the fourth may have arbitrary value, thus representing a single Dirac function upon the graph.

It is emphasized that *in a graph no relations exist between the weights put upon the branches and the weights put upon the nodes.* That is, a weighted graph—as a unit—is not aware of the existence of such physical laws as Kirchhoff's.

Graphs as frustrated electrical networks. Instead of the masses, let now an electrical current and voltage be impressed on the graph, as so many writers on the subject claim to do. Immediately a host of contradictions arise:

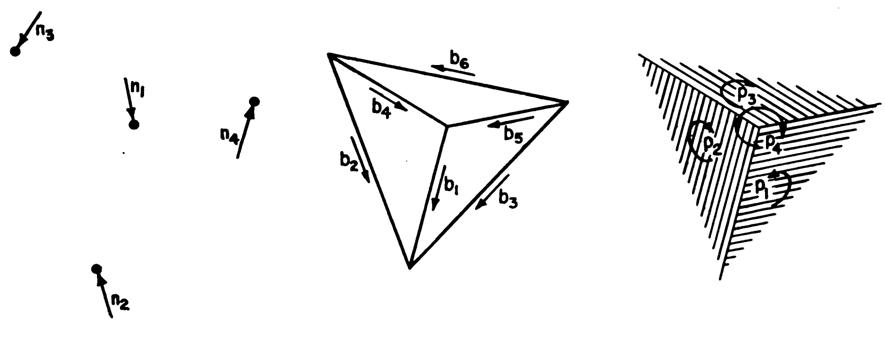
1) A current can not be placed on *one* node only, as a weight can be. The current would only accumulate and eventually explode.

2) If two currents are placed on two nodes, the two currents must be equal and also must have opposite signs. That is not true with the weights.

3) The summation of masses placed upon one node and its surrounding branches may be any number. On the other hand, the summation of all currents entering a node from the outside and its surrounding branches must be zero.

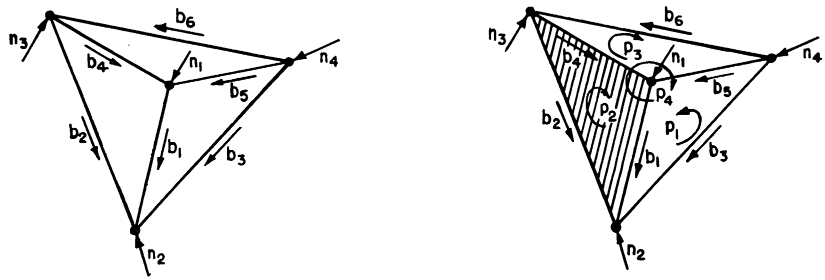
4) The summation of masses placed upon all the nodes of a graph may be any number (usually it is normalized to one), while the summation of currents entering simultaneously all nodes (the entire network) must be zero.

5) If a series generator is placed within a branch, *the network is severed. The number of nodes has thereby increased by one* (but not the number of branches) and *the graph has been radically changed into a different graph.* However, in a graph the number of nodes,



a) 0-NETWORK b) 1-NETWORK (BRANCH NETWORK) c) 2-NETWORK

FIG. 1. Three p-networks.



a) GRAPH b) PLANAR POLYHEDRON

FIG. 2. Two false polyhedral models of electrical networks.

as well as the number of branches, must remain invariant under any operation or transformation.

6) Any open-circuiting or short-circuiting operation is prohibited, since the number of nodes increases or decreases respectively.

No matter what the electrical engineer attempts to do with a graph, he is violating some basic property of graphs, or of electricity, or of both.

Three types of 1-networks. In any electrically excited network the simultaneous coexistence of three different types of branch-networks (or 1-networks) are assumed, each of which is, in general, *differently interconnected*. In particular:

1) An *abstract* network, whose branches form the geometrical "reference frame".

2) A *material* network, whose branches possess an impedance operator z.

3) An *electrical* network, whose branches are intangible, but physical filaments.

The "abstract" 1-network. A set of abstract branches are characterized by a set of unit vectors b_α, b^α and a square, nonsingular connection matrix $C^{\alpha}_{\alpha'}$ whose inverse is $A^{\alpha'}_{\alpha}$. They describe the manner of interconnection (or rather configuration) of branches. There are as many rows and columns in C and A, as there are branches. Both matrices contain in general only $+1$, -1 and 0 elements. (There exist also a set of unit matrices.)

Two types of independent sets of columns exist in C that enumerate the formation by the branches of two types of "chains" or "paths":

1) Apparently "open" chains (or "open-paths") described by a rectangular C_0.

2) Apparently "closed" chains (or "closed-paths") described by a second rectangular C_c.

The direct sum of C_0 and C_c is the square C. The inverse matrix $A = C_t^{-1}$ also splits into A^0 and A^c. The manner of selection of the two types of paths depends upon the problem at hand. For Fig. 1b a particular independent set of paths is

$$C^{(1)} = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{array} \begin{array}{c} \begin{array}{cccccc} o_1 & o_2 & o_3 & c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & \\ & & & & -1 & \\ & & & -1 & & \\ & 1 & & 1 & -1 & \\ & & 1 & 1 & & 1 \\ & & & & & -1 \end{array}\right] \end{array} = s_{(1)}\begin{array}{cc} o_{(1)} & c_{(1)} \\ {[C_0^{(1)}} & C_c^{(1)}] \end{array} \tag{1a}$$

The inverse connection matrix and its subdivisions are

$$A_{(1)} = \begin{array}{c} \\ b^1 \\ b^2 \\ b^3 \\ b^4 \\ b^5 \\ b^6 \end{array} \begin{array}{c} \begin{array}{cccccc} o^1 & o^2 & o^3 & c^1 & c^2 & c^3 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & -1 & \\ & & 1 & -1 & & \\ -1 & 1 & & & & \\ -1 & & 1 & & & \\ & -1 & 1 & & & -1 \end{array}\right] \end{array} = s^{(1)}[\overset{o^{(1)}}{A^0_{(1)}} \quad \overset{c^{(1)}}{A^c_{(1)}}] \qquad (1b)$$

Actually the abstract branches are isolated from each other, since there is nothing to indicate that they are interconnected. These matrices may assume, of course, special forms by special selection of the open and closed paths.

Orthogonality of abstract networks. Since A is the inverse of C_t, it follows that $C_tA = 1$ or, splitting C and A into their two rectangular components:

$$\begin{array}{ll} C_{0t}A^0 = 1 & C_{0t}A^c = 0 \\ C_{ct}A^0 = 0 & C_{ct}A^c = 1 \end{array} \qquad (2)$$

Thus the rectangular matrices C_c , A^0, as well as C_0 , A^c are orthogonal to each other. These relations originated the designation "orthogonal network", to represent a general network in which the open-paths and closed-paths are orthogonal to each other

The "electrical" 1-network. Imitating the flow of fluid in a region, two types of *permanently* connected current filaments ("blue" and "red" currents) are assumed to exist in any network:

1) *"Solenoidal"* currents, that flow in closed paths. They are denoted by lower case letters i.

2) *"Lamellar"* currents, that flow in open paths. They are denoted by capital letters I.

In the presence of these two types of flow-patterns, no current can accumulate at any node or within the network, since whatever current enters a node, the same current must depart from it. (A solenoidal current neither enters a network, nor departs from it.) Hence *Kirchhoff's current law* ($\sum I = 0$) *is automatically satisfied at each node and in the entire network,* without explicitly stating so.

The voltages are also called "solenoidal" e and "lamellar" E, but their physical representation is less visual.

Since voltages have *radically* different properties, physical and geometrical representations, or laws of transformation from currents, the voltages are "covariant" vectors e_α , E_α while the currents are "contravariant" vectors i^α, I^α.

The "material" 1-network. The material network (with conductances σ_e) is characterized by a set of "impedance" and "admittance" matrices $z_{\alpha\beta}$ and $y^{\alpha\beta}$. Their component elements, (say $Z_1 + Z_2 + Z_3$), describe, unequivocally, which material branches are interconnected in series or in shunt.

It is emphasized that the material and electrical networks are in general differently interconnected. That is, an "open" current may flow in a "closed" material path and vice versa.

Absence of connectedness. Moreover, during the analysis or during a network operation, a continuous "switching" action takes place in the material network. As the currents and voltages become zero, some open-circuits become short-circuited, and some short-circuits become open-circuited. That is, *the connectedness of the material network depends on the nature of the superimposed electrical signals, just as in any neural net* (artificial or natural).

It should be mentioned that changes occur also in the electrical networks. Here an open current or voltage may be *replaced* by a closed current or voltage, and vice versa.

Four extreme interconnections. The existence of the four types of rectangular connection tensors C_0, C_c, A^0, A^c indicates the existence of the following four types of *extreme* operating (or switching) conditions in any branch-network (Fig. 3).

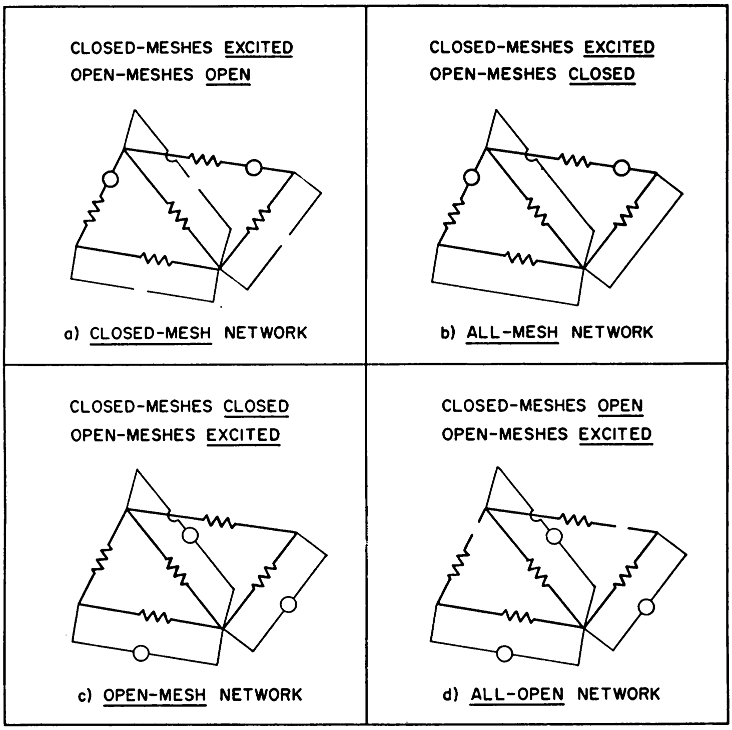


FIG. 3. Four extreme interconnections of any branch-network.

1) The closed paths are excited and the open paths are left *open* ("mesh" networks). They are constructed with C_c.

2) The closed paths are excited and the open paths are *short-circuited* ("all-mesh" networks). They are constructed with A^c.

3) The open paths are excited and the closed paths are left *closed* ("junction "or "open-path" networks). They are constructed with A^0.

4) The open paths are excited and the closed paths are *open-circuited* ("all open" networks). They are constructed with C_0.

In these switching operations *the number of nodes in general does not remain invariant, but the number of branches always does.* In a graph the requirement of "invariance of the number of nodes" prevents all types of switching operations, except the open-path type. Even in forming a mesh network each generator tears the network and increases the number of nodes.

Influence of "environment". In algebraic topology with every n-dimensional polyhedron there is associated a "dual" polyhedron, whose $n - p$ simplexes are *at right angles in space* to the n-simplexes of the original (primal) polyhedron. All analysis of surfaces imbedded in multiply-connected spaces imply the use of both polyhedra. Similarly it so happens that it is necessary to employ both primal and dual polyhedra (a "two-phase" system) to propagate a magneto-hydrodynamic wave across them.

Therefore an isolated 1-network with four electrical parameters e, i, E, I is not yet a "complete" network that could serve as a prototype for a polyhedral network. It is necessary to consider the *environment* of the 1-network that can induce currents and voltages in the network under consideration. *Two-phase electrical power-networks and rotating electrical machines are models of such an environment.* The topological organization of "complete" dynamic 1-networks (immersed into a two-dimensional magnetic field) has been undertaken by the author in his many works on non-Riemannian dynamics (Ref. 3). The coupled 1-networks are also tearable and orthogonal. However, the organization of single-phase *static* 1-networks (with which the present paper deals) is only a necessary preliminary step to the organization of two-phase *dynamic* 1-networks. It is the latter that form the real foundation of polyhedral networks excited with electromagnetic waves.

The external electrical 1-network also possesses the four parameters e^*, i^*, E^*, I^*. However when these appear as *induced* parameters in the original network, they behave in a dual manner. In particular:

1) In the closed paths of e and i appear the open parameters E^* and I^*.

2) In the open paths of E and I appear the closed parameters e^* and i^*.

Thus in any electrical 1-network (or p-network) eigh types of electrical parameters co-exist simultaneously, namely four covariant vectors e, e^*, E, E^* and four contravariant vectors i, i^*, I, I^*. (The existence of mechanical, thermodynamical and other types of parameters in 1-networks can also be assumed. Their discussion, however, is left for future considerations.)

Eight electromagnetic parameters. Although most electrically excited 1-networks contain the e, i, $z = 1/\sigma_e$ parameters, there arise occasionally dielectric 1-networks with D, E, ϵ, and magnetic 1-networks with B, H, μ parameters. The theoretical existence of magnetic conduction-networks with E_m, I_m, σ_m also must be assumed. When these four types 1-networks are coupled, the eight electrical parameters may be reinterpreted as forming the parameters of an electromagnetic wave. In particular:

i = electrostatic flux d	e = electric intensity e
I^* = electric charges ρ^{e*}	E^* = magnetic current J_m^*
I = electric current J^e	E = magnetic charge ρ_m
i^* = magnetic intensity h^*	e^* = magnetic flux b^*

(The dots over b, d, ρ^e, ρ_m are left out because of printing exigencies.)

It is interesting that the closed (solenoidal) parameters represent *wave* quantities b, h, d, e, while the open (lamellar) parameters represent *particles* ρ^e, ρ_m, J^e, J_m. Such a set of four coupled 1-networks has already been used as the three-dimensional 1-*network* model (not polyhedral model) of the field equations of Maxwell (Ref. 5). This three-dimensional model serves as the *prototype* for each complete wave—within the hierarchy of electromagnetic waves—that propagates across an n-dimensional polyhedron.

It should be mentioned that in the polyhedral model the above eight parameters of an electromagnetic wave appear, not in their eight *vectorial* forms shown, but in their four *tensorial* forms $F_{\alpha\beta}$, $H^{\alpha\beta}$, s_α and s^α. It is again found that the three concepts of multi-dimensional networks, electricity and tensors are all intertwined into one consistent structure, just as they are in the case of conventional electrical networks.

Signal-flow diagram. All tensorial formulas of a 1-network may be codified in the form of an algebraic diagram, or signal-flow graph, (a true "graph", Fig. 4) in which each arrow represents a tensor instead of a scalar. More detailed signal-flow diagrams may be found in Ref. 2.

Generalization to *p*-networks. Every concept, relation, switching operation, algebraic diagram, formula, etc, developed for 1-networks, are valid also for any other

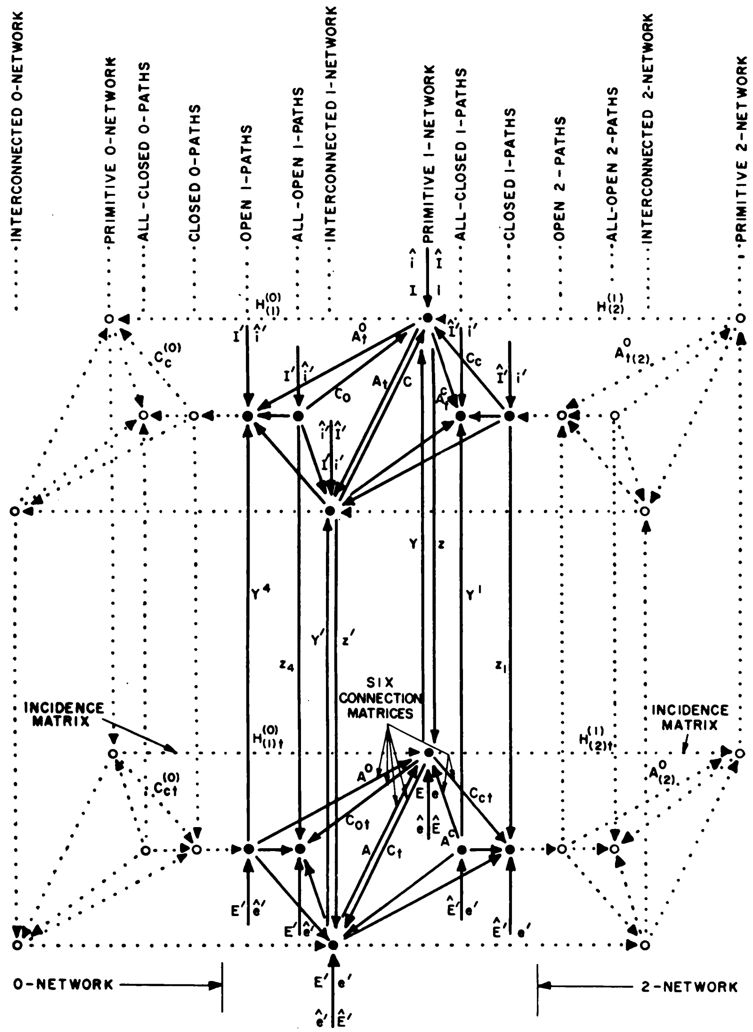


FIG. 4. Signal-flow diagram of a branch-network.

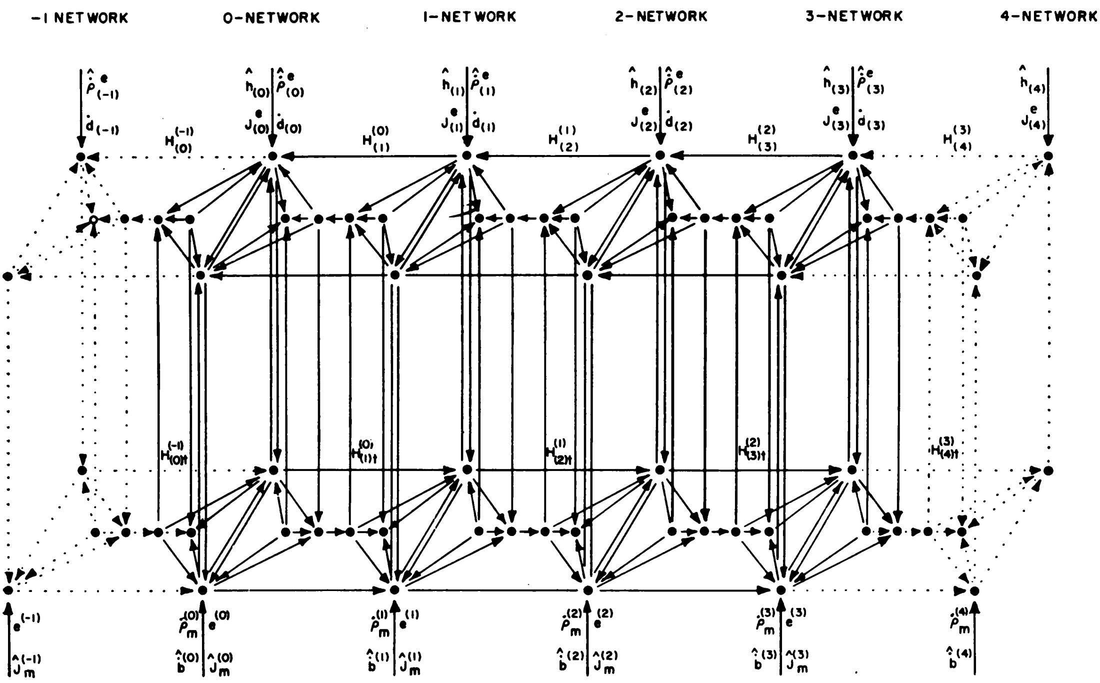


FIG. 5. Signal-flow diagram of a polyhedron.

p-networks, namely for 0-networks, 2-networks, etc. The same nomenclature is employed for each p-network, but with a dimension-index p attached. It is possible to assume that each p-network can be excited separately by an electrical e, i, z system as long as they are not interconnected.

To prepare the ground for the interconnection of the various p-networks into a polyhedron, the square connection matrices (and their two rectangular subdivisions) will be given for a 0-network and for a 2-network. They are the two neighbors of the electrical 1-network and serve as examples for the above "generalization postulate".

Connection matrices of a 0-network. Let the 0-network of Fig. 1a with 4 points be considered. Its "closed" paths are formed by any two vertices ("node-pairs"). Altogether there are three such closed paths. Its single "open" path consists of the collection of all vertices. Thus the square nonsingular connection-matrix $C^{(0)}$ of a 0-network (and its two rectangular subdivisions) are

$$C^{(0)} = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{o}_p & \mathrm{c}_a & \mathrm{c}_b & \mathrm{c}_c \end{array} \\ \left[\begin{array}{c|ccc} 1 & 1 & & \\ 1 & -1 & -1 & -1 \\ 1 & & 1 & \\ 1 & & & 1 \end{array}\right] \end{array} = s_{(0)}\begin{array}{c} \begin{array}{cc} \mathrm{o}_{(0)} & \mathrm{c}_{(0)} \end{array} \\ [C_0^{(0)} \quad C_c^{(0)}] \end{array} \qquad (3)$$

The three closed-paths of the 0-network were assumed as the boundary points of the three open-paths of the 1-network. It may come as a surprise to many graph theorists that a node-pair is a "closed" concept (analogous to a "closed-mesh") and it is not an "open" concept (like the "open 1-path" is).

The inverse connection matrix and its two subdivisions are

$$A_{(0)} = \begin{array}{c} \\ n^1 \\ n^2 \\ n^3 \\ n^4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{o}^p & \mathrm{c}^a & \mathrm{c}^b & \mathrm{c}^c \end{array} \\ \left[\begin{array}{c|ccc} 1 & 3 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{array}\right] \end{array} \tfrac{1}{4} = s^{(0)} \begin{array}{c} \begin{array}{cc} \mathrm{o}^{(0)} & \mathrm{c}^{(0)} \end{array} \\ [A^0_{(0)} \quad A^c_{(0)}] \end{array} \tag{4}$$

Since $C_t^{(0)} A_{(0)} = 1$, the orthogonal relations shown in Eq. 2 are satisfied in a 0-network also.

Connection matrices of a 2-network. Let the 2-network of Fig. 1c with 4 planes be considered. (The fourth plane is the sum of the visible 3 planes.) The "open paths" are formed by those planes that cover the "closed paths" of the 1-network of Fig. 2b. The square connection matrix (and its two rectangular subdivisions) are

$$C^{(2)} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{o}_A & \mathrm{o}_B & \mathrm{o}_C & \mathrm{c}_P \end{array} \\ \left[\begin{array}{ccc|c} 1 & & & -1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & 1 \end{array}\right] \end{array} = s_{(2)} \begin{array}{c} \begin{array}{cc} \mathrm{o}_{(2)} & \mathrm{c}_{(2)} \end{array} \\ [C_0^{(2)} \quad C_c^{(2)}] \end{array} \tag{1a}$$

The inverse connection matrix and its rectangular subdivisions are

$$A_{(2)} = \begin{array}{c} \\ p^1 \\ p^2 \\ p^3 \\ p^4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{o}^A & \mathrm{o}^B & \mathrm{o}^C & \mathrm{c}^P \end{array} \\ \left[\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & -1 & -1 & 1 \end{array}\right] \end{array} = s^{(2)} \begin{array}{c} \begin{array}{cc} \mathrm{o}^{(2)} & \mathrm{c}^{(2)} \end{array} \\ [A^0_{(2)} \quad A^c_{(2)}] \end{array} \tag{5b}$$

Since $C_t^{(2)} A_{(2)} = 1$, the relations in Eq. (2) are satisfied and the 2-network is also an "orthogonal" network.

Connection matrices and Kirchhoff's laws. In each p-network the solenoidal and lamellar currents and voltages separately satisfy Kirchhoff's laws (for p-currents and p-voltages) in consequence of the properties of the four rectangular connection matrices C_0, C_c and A^0, A^c of each p-network.

From the algebraic diagram of Fig. 4, four sets of Kirchhoff's laws are now derived *by inspection*. The first two sets relate to the original four vectors E, i and e, I respectively

$$\begin{array}{r|l} C_{ct}^{(p)} E_{(p)} = 0 & A^0_{(p)t} i^{(p)} = 0 \\ C_{0t}^{(p)} e_{(p)} = 0 & A^c_{(p)t} I^{(p)} = 0 \end{array} \tag{6}$$

A second two sets relate to the environmental vectors. (See Ref. 2, Part XXIX Eqs. 18 and 19.)

Interconnected polyhedron. A sequence of $n + 1$ different p-networks, consisting of a 0-network, 1-network, 2-network etc, to n-network, can be connected into an n-dimensional polyhedron. The algebraic or signal-flow diagram of the resultant structure (Fig. 5) consists of the signal-flow diagram of each component p-network (given in Fig. 4) laid side-by-side. Because of the interconnections additional arrows appear between the dots, some of which are shown on Fig. 5. The most important of these new tensors is the so-called "incidence matrix" M^p_{p+1} (long horizontal arrows) that shows how two neighboring sets of p-simplexes and $p + 1$ simplexes are interconnected.

Polyhedra excited with electromagnetic waves can be used for multidimensional curve-fitting (Ref. 13), partial differential equations, Fourier Transform, and advanced information-theoretical problems, where a large number of useful informations need to be derived from a comparatively few given data.

Incidence matrix of a graph. In the signal-flow diagram of Fig. 5 the rectangular incidence matrix M^0_1 is shown to relate the points of the 0-network with the branches of the 1-network. The diagram also shows that M^0_1 *may be established as the product of two rectangular connection matrices* $C^{(0)}_c$ *and* $A^{(1)}_0$, *one belonging to the* 0-*network, the other to the* 1-*network.* That is (Fig. 2a)

$$C_c^{(0)} 1_0^c A^0_{(1)t} = C_c^{(0)} A^0_{(1)t} = M^{(0)}_{(1)} \tag{7a}$$

$$\begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array}\begin{array}{c} \begin{array}{ccc} c_a & c_b & c_c \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ -1 & -1 & -1 \\ & 1 & \\ & & 1 \end{array}\right] \end{array} \begin{array}{c} \\ o^1 \\ o^2 \\ o^3 \end{array}\begin{array}{c} \begin{array}{cccccc} b^1 & b^2 & b^3 & b^4 & b^5 & b^6 \end{array} \\ \left[\begin{array}{cccccc} 1 & & & -1 & -1 & \\ & 1 & & 1 & & -1 \\ & & 1 & & 1 & 1 \end{array}\right] \end{array} = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array}\begin{array}{c} \begin{array}{cccccc} b^1 & b^2 & b^3 & b^4 & b^5 & b^6 \end{array} \\ \left[\begin{array}{cccccc} 1 & & & -1 & -1 & \\ 1 & -1 & -1 & & & \\ & 1 & & 1 & & -1 \\ & & 1 & & 1 & 1 \end{array}\right] \end{array} \tag{7b}$$

In topology the incidence matrix (or rather its abstract equivalent) is denoted by ∂ and δ. Customarily M^0_1 is established by inspection of the nodes and branches. The columns of M^0_1 show that each branch extends between two nodes. It is very important to note that *a connection matrix can not be derived from an incidence matrix,* in spite of the sleight-of-hand manipulations of graph theorists. On the other hand, an incidence matrix can be found as the product of two rectangular connection matrices.

Incidence matrix of a planar polyhedron. If a planar polyhedron is defined as a collection of nodes, branches and planes, then the second of its incidence matrices is found by (Fig. 2b)

$$C_c^{(1)} 1_0^c A^0_{(2)t} = C_c^{(1)} A^0_{(2)t} = M^{(1)}_{(2)} \tag{8a}$$

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{array}\begin{array}{c} \begin{array}{ccc} c^1 & c^2 & c^3 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & \\ & -1 & \\ -1 & & \\ & 1 & -1 \\ 1 & & 1 \\ & & -1 \end{array}\right] \end{array} \begin{array}{c} \\ o^A \\ o^B \\ o^C \end{array}\begin{array}{c} \begin{array}{cccc} p^1 & p^2 & p^3 & p^4 \end{array} \\ \left[\begin{array}{cccc} 1 & & & 1 \\ & 1 & & -1 \\ & & 1 & -1 \end{array}\right] \end{array} = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{array}\begin{array}{c} \begin{array}{cccc} p^1 & p^2 & p^3 & p^4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & & \\ & -1 & & 1 \\ -1 & & & -1 \\ & 1 & -1 & \\ 1 & & 1 & \\ & & -1 & 1 \end{array}\right] \end{array} \tag{8b}$$

The columns of M_2^1 show that each plane extends between three branches. In general a planar polyhedron is completely represented by the *three* nonsingular connection matrices $C^{(0)}$, $C^{(1)}$ and $C^{(2)}$.

Incidence matrices and Stokes' Theorem. When two different-dimensional spaces, or networks of spaces, are interconnected, all superimposed scalar, vector and tensor quantities Φ must satisfy Stokes' Theorem at the boundaries. For instance, if c is a $p + 1$ volume and ∂c is its boundary p-surface, then

$$\int_{\partial c} \Phi = \int_c d\Phi \tag{9a}$$

That is, the surface integral of a tensor Φ is equal to the volume integral of the "exterior" differential ($d\Phi$) of the tensor.

Stokes' Theorem does not apply to a single p-network, hence it can not apply to an electrical network. When two different-dimensional networks are interconnected, an electro magnetic wave (or rather a magnetohydrodynamic wave) must be passed across them. The electromagnetic wave vectors do satisfy Stokes' Theorem.

The *incidence matrices,* since they relate two different-dimensional electromagnetic quantities, *automatically satisfy Stokes' theorem* for both solenoidal and lamellar vectors, as well as for both covariant and contravariant vectors. E.g. from Fig. 5

$$\begin{array}{ll|ll} E_{p+1} &= (M^p_{p+1})_t\, B_p & H^p &= M^p_{p+1}\, D^{p+1} \\ b_{p+1} &= (M^p_{p+1})_t\, e_p & d^p &= M^p_{p+1} h^{p+1} \end{array} \tag{9b}$$

Since no graph-theorist or polyhedral-theorist of electrical networks ever mentions Stokes' theorem, although they claim to use incidence matrices, this silence also throws doubt upon the correctness of their models. Confusing incidence matrices with connection matrices is identical to confusing Stokes' theorem with Kirchhoff's laws.

Transforming graphs into instruments of torture. It can be stated that no electrical network problem exists which can not be analyzed and solved by the use of branches only. Every attempt to drag in nodes or planes must lead to disaster or to a forced (but camouflaged) switch-back to the branches somewhere during the reasoning process Expressed in another manner, the final equations of state of any electrical network as derived by graph-theorists, can always be arrived-at in a more logical manner, an in considerably fewer steps, if the analysis starts outright with the presence of branches only, and if *the concept of "node" is never mentioned.* Moreover, the engineer would not have tied himself meanwhile into knots, and would not have been reduced to intellectual impotency by the many restrictions and "verboten" signs scattered along the pathways of his reasoning by such graph-theoretical concepts as the "invariance of the number of nodes." On the other hand, the 1-network restriction of "invariance of the number of branches" would still enable the engineer to perform all switching, tearing, and other operations and transformations customarily performed at all times on all varieties of electrical networks.

Tensors in the large. By calling a tensor a "matrix", the electrical engineer thinks that he has removed thereby the crucial difficulties of his analysis. But he has merely stuck his head under the sand! He has ignored the fact that each scalar number he deals with is actually a p-volume integral of a differential p-form, whose definition implies tensors. Moreover he has deliberately removed the only mechanism he had available to differentiate between the numerous types of reference frames he is forced to deal with.

One future course for the development of electrical engineering suggests the extension of one-dimensional networks to multi-dimensional networks: and the generalization of conventional currents to electromagnetic and magnetohydrodynamic waves (to both ray-optics and wave-optics). An obvious possibility of physically realizing such structures lies in the use of *crystals*, since the oscillating atoms in a polyatomic molecule, as well as their associated dipole waves and electromagnetic waves, are surprisingly comprehensive physical analogues of the polyhedral waves discussed here. (The atoms within a molecule define the vertices of a polyhedron.) To facilitate these structural researches in *solid-state physics*, the electrical engineer will definitely need the theory of "tensors in the large" as developed during the last three decades by Cartan, DeRham, Hodge, Whitney, Eilenberg, Steenrod, Chern, Lichnerowitz, Kodaira, and a host of other mathematicians. Only the recently appearing advanced textbooks on differential topology ("fiber bundles", etc.) contain the tensorial and topological foundations upon which electrical engineers of the future must build their electrified polyhedral structures.

Algebraic Topology—A false alarm for electrical networks. It is important to note that texts on algebraic topology superimpose upon the underlying structure only such geometrical concepts which can not serve as analogues for electrical entities. The geometrical analogues of electricity are represented mathematically as "*exterior*" *differential forms* and their p-volume integrals. Only the above mentioned recent textbooks on *differential* topology deal with the type of differential forms needed by electrical engineers. Conventional texts on *algebraic* topology do not deal with superimposed differential forms or their integrals.

Moreover, electricity and Maxwell's equations constantly require the appearance and application also of Stokes' theorem, and the latter does form the central interest in differential topology. On the other hand, textbooks on algebraic topology never mention Stokes' theorem. Hence the theorems in the latter texts on superimposed concepts cannot be forced by any strategy to apply to electrical networks.

The Concepts of "Tearing and Interconnecting". It should be emphasized that the concept of "tearing and interconnecting" (stating that all p-networks with n p-simplexes can be transformed into each other) is indispensable to the practical applications of electrified 1-networks and p-networks. However, the concept—or at least a systematic treatment of it—is believed to be absent not only in texts on algebraic topology, but also in texts on differential topology. That fact, however, does not make the invariant procedure of tearing and interconnecting scientifically incorrect. Impulse functions also appeared first in physics and then only in works on mathematics.

Crystal Computers. With the arrival of multidimensional networks the eventual construction of "crystal computers" (both analogue and digital) is within the realm of possibility. Since the dynamically self-sustained dipole and more general waves in crystals can be used to model polyhedral self-organizing automata, that fact opens up new horizons for the theoretical and practical solution of large-scale system problems. It is also probable, that the electromagnetic waves in a crystal approximate much closer the actual phenomena taking place within a neural network, than do the signals in a switching circuit. While the possibilities for the utilization of multidimensional networks are limitless, much organizational and development work still lies ahead.

But topologists can not be expected to do the highly specialized research demanded by the vagaries of electricity. Electrical engineers should not thus copy and adapt blindly everything the algebraic topologists puts down on paper, or tells them in casual conversations. Rather they should do their own share of the topological research in their own bailiwicks.

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