NOTE ON GREEN'S FUNCTION IN ANISOTROPIC ELASTICITY*

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Abstract. Green's function for an elastic, anisotropic medium is constructed with the help of the method used by Courant and Hilbert in the study of light propagation through an anisotropic dielectric. The amplitudes of the arrivals corresponding to each wave surface are expressed as functions of the curvature of the normal surfaces. Multi-valued solutions are expressed as integrals over the dislocation lines, whose motion is assumed to be known, but otherwise arbitrary. The formulae are worked out in detail for the isotropic medium.

Introduction. The purpose of this note is to put on record a number of formulae which have been derived with the help of Green's function for an anisotropic elastic medium. Green's function can be constructed formally by a method which is described by Courant and Hilbert [1]. The resulting expression can be used directly to investigate its main singularities, called the arrivals. Also Green's integral theorem yields solutions of the equations of elasticity which can be interpreted as describing dislocations in arbitrary motion.

Since the emphasis of the present work is on finding formulae for the arrivals and the multivalued displacements, no particular effort has been made to establish general, but still sufficient conditions for the validity of these results. For the derivation of Green's function a reference to the more recent work on distributions, e.g. Lighthill [2], may be sufficient. For the construction of multivalued displacements the existence of the necessary continuity in the dislocation shapes and motions has been tacitly assumed. In order to illustrate the general procedure, the case of an isotropic medium has been discussed in more detail. In particular it is shown in which manner the multivaluedness arises in the integral representation of the dislocation.

1. Symbols and basic formulae. Greek indices run from 0 to 3. Latin indices run from 1 to 3. If an index occurs twice in a product, it has to be summed. Space coordinates are (x_1, x_2, x_3) or (x, y, z), the time coordinate is x_0 or t. The displacement is given by a vector (u_1, u_2, u_3) or (v_1, v_2, v_3), the body forces by a vector $\rho(f_1, f_2, f_3)$ or $\rho(g_1, g_2, g_3)$, where the density ρ of the elastic medium has been inserted for convenience. Finally there is the stress tensor σ_{ij} and the strain tensor ϵ_{ij}.

The unbounded anisotropic medium is now described in the ordinary fashion by the following relations:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \tag{1}$$

$$\sigma_{ij} = c_{ij,kl}\epsilon_{kl} = c_{ij,kl}\frac{\partial u_k}{\partial x_l}; \qquad \text{(Hooke's law)} \tag{2}$$

$$c_{ij,kl} = c_{ij,lk} = c_{ji,lk}; \qquad c_{ij,kl} = c_{kl,ij}; \tag{3}$$

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$$\rho \frac{\partial^2 u_k}{\partial t^2} = \rho f_k + \frac{\partial \sigma_{kl}}{\partial x_l} ; \qquad \text{(Conservation of momentum)} \tag{4}$$

$$\gamma_{ij,\kappa\lambda} \frac{\partial^2 u_j}{\partial x_\kappa \, \partial x_\lambda} = -f_i ; \qquad \text{(Equations of motion)} \tag{5}$$

$$\gamma_{ij,00} = -\delta_{ij} ; \qquad \gamma_{ij,0k} = \gamma_{ij,k0} = 0; \tag{6}$$

$$\gamma_{ij,kl} = \frac{1}{\rho} c_{ik,jl} ; \qquad \text{(note the change in index order)} \tag{6}$$

The symmetry of the coefficients γ implies the formula

$$u_i g_i - v_i f_i = \frac{\partial}{\partial x_\kappa} \left(v_i \gamma_{ij,\kappa\lambda} \frac{\partial u_j}{\partial x_\lambda} - u_i \gamma_{ij,\kappa\lambda} \frac{\partial v_j}{\partial x_\lambda} \right), \tag{7}$$

where

$$\gamma_{ij,\kappa\lambda} \frac{\partial^2 u_j}{\partial x_\kappa \, \partial x_\lambda} = -f_i , \qquad \gamma_{ij,\kappa\lambda} \frac{\partial^2 v_j}{\partial x_\kappa \, \partial x_\lambda} = -g_i .$$

Green's integral formula results from (7) by integrating both sides over some volume in (t, x, y, z) space and transforming the righthand side into a surface integral.

2. Construction of Green's function. Green's function for the equations of elasticity can be obtained with the help of the method which is described by Courant and Hilbert [1], (p. 260 ff., 460 ff.). Since the procedure is straightforward, there is no need for explaining the details; but the result will be stated explicitly.

Let us consider a matrix of 3 rows and 3 columns whose elements m_{ij} are quadratic functions of the four variables ξ_0, ξ_1, ξ_2, ξ_3 given by

$$m_{ij} = \gamma_{ij,\kappa\lambda} \, \xi_\kappa \xi_\lambda . \tag{8}$$

If M is the determinant $| m_{ij} |$, and M_{ij} the minor of m_{ij}, then M is of sixth, and M_{ij} is of fourth order in ξ_μ. We define now differential operators D and D_{ij} which are obtained from M and M_{ij} by formally replacing $\xi_\mu \rightarrow \partial/\partial x_\mu$. These operators satisfy the relation

$$\gamma_{ij,\kappa\lambda} \, \partial^2/\partial x_\kappa \, \partial x_\lambda \cdot D_{jk} = \delta_{ik} \cdot D. \tag{9}$$

A polynomial E of sixth order in the variables ξ, η, ζ can be derived from the polynomial M in the variables ξ_μ by letting $\xi_0 = 1$, $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \zeta$. Similarly polynomials E_{ij} of fourth order in the variables ξ, η, ζ are derived from M_{ij}. The equation $E = 0$ determines in general 3 surfaces in the space of coordinates (ξ, η, ζ), which are called the normal surfaces. For a given direction in (ξ, η, ζ) space the distances from the origin to these three surfaces are called $1/a$, $1/b$, $1/c$, in increasing order.

The kernel associated with the sixth-order differential operator D is then formally given as an integral over the normal surfaces

$$K = (2\pi)^{-3} \iint \left(1 - \frac{b^2}{a^2}\right)^{-1} \left(1 - \frac{c^2}{a^2}\right)^{-1} (\boldsymbol{\xi} \, d\boldsymbol{\Sigma}) \int_0^\infty \frac{ds}{s^3} \sin st \cos s \boldsymbol{\xi} \cdot \mathbf{x} + \text{c.p.}, \tag{10}$$

where $d\boldsymbol{\Sigma}$ is the surface element of the a-normal surface, and the cyclic permutation refers to permuting the triple (a, b, c) and the corresponding normal surfaces. The vectors $\boldsymbol{\xi}$ and $\mathbf{x}$ are abbreviations for (ξ, η, ζ) and (x, y, z). $\boldsymbol{\xi} \cdot \mathbf{x}$ is the scalar product.

Green's function for the equations of elasticity (5) follow now from (9) and (10) by differentiation

$$K_{ij} = \frac{\partial^2}{\partial t^2}\left\{(4\pi)^{-2} \iint \left(1 - \frac{b^2}{a^2}\right)^{-1}\left(1 - \frac{c^2}{a^2}\right)^{-1} E_{ij}\ \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma} + \text{c.p.}\right\}. \tag{11}$$

For given values of the coordinates (x, y, z) and the time t the integration is taken over those parts of the normal surfaces which are contained between the planes $\boldsymbol{\xi}\cdot\mathbf{x} = t$ and $\boldsymbol{\xi}\cdot\mathbf{x} = -t$ in (ξ, η, ζ) space. All singular integrations or differentiations are to be interpreted with the help of the theory of generalized functions [2].

K_{ij} gives the displacement at the point (x, y, z) and time t in the i-direction due to an impulse of unit strength in the j-direction which is concentrated in space at the origin and in time at $t = 0$. K_{ij} can be shown to vanish for small t and for large t, namely if t is so small that the planes $|\boldsymbol{\xi}\cdot\mathbf{x}| = t$ intersect all normal surfaces, and if t is so large that the planes $|\boldsymbol{\xi}\cdot\mathbf{x}| = t$ intersect none of them. For given coordinates (x, y, z) Green's function K_{ij} has a singularity whenever the planes $|\boldsymbol{\xi}\cdot\mathbf{x}| = t$ are tangent to one of the normal surfaces. If the singularity is a δ-function, one can call this singularity an arrival since it may be compared with the signal propagation by compressional waves in a liquid.

3. The amplitudes of the arrivals. Since K_{ij} depends mainly on the direction of the vector (x, y, z) and on the reduced time $\tau = t/r$ with $r = (x^2 + y^2 + z^2)^{1/2}$, Green's function can also be written as

$$K_{ij} = (4\pi r)^{-2} \frac{\partial^2}{\partial \tau^2}\left\{\iint E_{ij}\left(1 - \frac{b^2}{a^2}\right)^{-1}\left(1 - \frac{c^2}{a^2}\right)^{-1} \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma} + \text{c.p.}\right\} \tag{12}$$

with the condition $|\boldsymbol{\xi}\cdot\mathbf{x}| < \tau r$ for the integration over the normal surfaces.

For a fixed point (x, y, z) we consider a value τ^0 such that the planes $|\boldsymbol{\xi}\cdot\mathbf{x}| = \tau^0 r$ are tangent to one of the normal surfaces (called the critical surface to distinguish it from the other normal surfaces) at the points $\pm(\xi^0, \eta^0, \zeta^0)$. It is convenient to use a local coordinate system $(\rho, \sigma, \tau - \tau^0)$ which is given by

$$\begin{aligned} \xi &= \xi^0 + \alpha_\xi \rho + \beta_\xi \sigma + \frac{x}{r}(\tau - \tau^0), \\ \eta &= \eta^0 + \alpha_\eta \rho + \beta_\eta \sigma + \frac{y}{r}(\tau - \tau^0), \\ \zeta &= \zeta^0 + \alpha_\zeta \rho + \beta_\zeta \sigma + \frac{z}{r}(\tau - \tau^0). \end{aligned} \tag{13}$$

The unit vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are tangent to the normal surface at (ξ^0, η^0, ζ^0), and directed along the directions of principal curvature for the normal surface through (ξ^0, η^0, ζ^0). The equation of the normal surface is then given by

$$\tau - \tau^0 = \frac{1}{2}\left(\frac{\rho^2}{R_1} + \frac{\sigma^2}{R_2}\right) + \cdots \tag{14}$$

in the neighborhood of the point of contact, R_1 and R_2 being the principal radii of curvature of the normal surface. The higher terms in the expansion of the equation for the normal surface will not be needed to find the amplitudes of the arrivals.

The integral over the normal surface is differentiated with respect to τ, so that its integrand has to be computed only in the neighborhood of the point (ξ^0, η^0, ζ^0). Indeed for an elliptic point the intersection of the planes $|\boldsymbol{\xi}\cdot\mathbf{x}| = \tau r$ with the critical surface shrinks as τ approaches τ^0. This is not true for a hyperbolic point, but then the intersection can be considered in only a limited fixed neighborhood, because the remaining part does not contribute to the singularity. In the same manner the intersections with the other (not critical) normal surfaces do not contribute to the singularity and are therefore neglected in the present investigation. It is further assumed that in the neighborhood of interest the functions E_{ij}, $(1 - b^2/a^2)$, $(1 - c^2/a^2)$, etc. are sufficiently well behaved. The singularities come then only from the peculiar behavior of the integration $\iint \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma}$ in the neighborhood of $\tau = \tau^0$. This procedure gives the strongest term in the singularities correctly.

At an elliptic point with $R_1 > 0$ and $R_2 > 0$ the approximate equation (14) can be solved by

$$\rho = [2R_1(\tau - \tau^0)]^{1/2}\cos\psi + \cdots, \qquad \sigma = [2R_2(\tau - \tau^0)]^{1/2}\sin\psi + \cdots, \tag{15}$$

so that the integration becomes

$$\iint \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma} = \tau^0(R_1R_2)^{1/2}\int_{\tau^0}^{\tau} d\tau \int_0^{2\pi} d\psi + \cdots$$
$$= 2\pi\tau^0(\tau - \tau^0)(R_1R_2)^{1/2} + \cdots, \qquad \tau > \tau^0. \tag{16}$$

An analogous result holds for $R_1 < 0$ and $R_2 < 0$

$$\iint \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma} = -\tau^0(R_1R_2)^{1/2}\int_{\tau}^{\tau^0} d\tau \int_0^{2\pi} d\psi + \cdots$$
$$= -2\pi\tau^0(\tau^0 - \tau)(R_1R_2)^{1/2} + \cdots, \qquad \tau < \tau^0. \tag{17}$$

Outside of the range for $\tau - \tau^0$ which is fixed by the sign of R_1 and R_2 at an elliptic point, the integrals above have to vanish.

At a hyperbolic point with $R_1 > 0 > R_2$ the equation (14) is modified by writing

$$\rho = \sqrt{R_1}\,(\rho^* + \sigma^*), \qquad \sigma = (-R_2)^{1/2}(\rho^* - \sigma^*), \qquad \tau - \tau^0 = 2\rho^*\sigma^* + \cdots. \tag{18}$$

The range of integration is now limited by some arbitrary limits for the variables ρ^* and σ^*, namely $-\rho^0 < \rho^* < \rho^0$ and $-\sigma^0 < \sigma^* < \sigma^0$. This gives

$$\iint \boldsymbol{\xi}\cdot d\boldsymbol{\Sigma} = \tau^0(-R_1R_2)^{1/2}\iint d\rho^*\cdot u\sigma^* + \cdots$$
$$= \tau^0(-R_1R_2)^{1/2}(\tau - \tau^0)\left\{\log\frac{2\rho^0\sigma^0}{|\tau - \tau^0|} + 1\right\} + \cdots, \tag{19}$$

where there is no restriction on the sign of $\tau - \tau^0$.

The differentiation with respect to t of (11) can now be performed on the approximate expressions (16) and (19), if one replaces again τ by t/r. K_{ij} is then to be understood as a function of t for a fixed point (x, y, z). The strongest singularity of K_{ij} for t in the neighborhood of $\tau^0 r$, i.e. the amplitude of the arrival, is then obtained at an elliptic point as

$$E_{ij}\left(1 - \frac{b^2}{a^2}\right)^{-1}\left(1 - \frac{c^2}{a^2}\right)^{-1}\frac{(R_1R_2)^{1/2}\tau^0}{4\pi r}\,\delta(t - \tau^0 r), \qquad \text{(or c.p.)} \tag{20}$$

and at a hyperbolic point as

$$E_{ij}\left(1 - \frac{b^2}{a^2}\right)^{-1}\left(1 - \frac{c^2}{a^2}\right)^{-1} \frac{(-R_1R_2)^{1/2}\tau^0}{4\pi r}\cdot\frac{1}{t - \tau^0 r}, \qquad \text{(or c.p.)} \tag{21}$$

These formulae have been written as if they referred to the a-normal surface as the critical surface. For the other surfaces as critical a cyclic permutation has to be made. The arguments of the polynomial E_{ij} are the coordinates (ξ^0, η^0, ζ^0) of the point of contact for the planes $|\, \boldsymbol{\xi}\mathbf{x} \,| = t$.

4. Multivalued displacements. Multivalued displacements can be defined in the following manner: A sequence of simply connected curves C_t in the plane $z = 0$, given by their coordinates $x^*(\sigma, t)$ and $y^*(\sigma, t)$ as functions of a parameter σ, is assumed to be known. The interior of C_t is defined as that part of the z-plane which lies to the left of C_t if one proceeds in the direction of increasing σ. The displacement v_i is continuous and has continuous derivatives outside of the plane $z = 0$. Also the derivatives of v_i are continuous across the z-plane for a given value of t, with the possible exception of the curve C_t. But the values of v_i are discontinuous across the z-plane at a given time t in such a manner that for a positive δ

$$\lim_{\delta\to 0}\, [v_i(t, x, y, -\delta) - v_i(t, x, y, +\delta)] = \begin{cases} \beta_i & \text{for } (x, y) \text{ inside } C_t, \\ 0 & \text{for } (x, y) \text{ outside } C_t. \end{cases} \tag{22}$$

Since the displacement v_i may be expected to have singularities on the curves C_t, a small cylinder Z_t around C_t has to be cut out of the (x, y, z) space at constant time t before applying Green's integral formula. Let $(l, m, 0)$ be the unit vector perpendicular to C_t for given value of σ, and let n be the speed with which the curve C_t moves in that direction for the value σ at the time t. A sequence of cylinders Z_t can then be defined by

$$t = t^* - \frac{\epsilon n}{\sqrt{1 + n^2}} \cos\vartheta, \qquad x = x^*(\sigma, t^*) + \frac{\epsilon l}{\sqrt{1 + n^2}} \cos\vartheta,$$

$$y = y^*(\sigma, t^*) + \frac{\epsilon m}{\sqrt{1 + n^2}} \cos\vartheta, \qquad z = \epsilon \sin\vartheta, \tag{23}$$

where the parameters t^*, σ, and ϑ are varied freely in order to describe a three dimensional surface in (t, x, y, z) space. ϵ is assumed to be a small positive quantity.

Green's integral theorem is now applied to the (t, x, y, z) space out of which the interior of the cylinders Z_t and a thin slab around the z-plane inside C_t have been cut out. This gives with Green's function (11) of the variables $t - t'$, $x - x'$, etc.

$$v_k(t, x, y, z) = -\int_{-\infty}^{t} dt' \int_{\text{interior } C_t} dx'\, dy' \beta_i \gamma_{ij,3\lambda} \frac{\partial K_{jk}}{\partial x'_\lambda} + \iiint_{Z_{t'}} d\Sigma'_\kappa \frac{\partial v'_i}{\partial x'_\lambda} \gamma_{ij,\kappa\lambda} K_{jk}. \tag{24}$$

The situation in the present problem is not stationary, and the second term does in general depend on the particular choice of the cylinders Z_t, even in the limit of small radius (small ϵ). This leads to an ambiguity in the definition of the multivalued displacement v_i which is connected with the possibility of applying concentrated forces on the curves C_t without changing the condition (22). Indeed such concentrated forces cannot (in the nonstationary case) be related unambiguously to the integral over the stresses acting on a small surface around the points of attack of these forces, as the cylinders Z_t around C_t. It can be shown that with the particular choice (23) of Z_t the second term of (24) vanishes in the limit $\epsilon \to 0$, if one inserts for v_i the function which is defined by the first term of (24). The displacement

$$v_k(t, x, y, z) = \beta_i \gamma_{ij,3\lambda} \frac{\partial}{\partial x_\lambda} \int_{-\infty}^{t} dt' \int dx'\, dy' K_{jk}(t - t', x - x', y - y', z - z') \tag{25}$$

where the last integral is extended over the interior of C_t, gives a multivalued solution of the equations of elasticity which describes a moving dislocation in the z-plane with the Burgers vector β_i. It is (at least apparently) independent of the particular choice of Z_t. At distances from C_t which are small compared to the radius of curvature of C_t, the displacement (25) corresponds to a uniformly moving dislocation.

The proof that v_i as defined by (25) has the correct discontinuities in the z-plane, whereas its derivatives are continuous, is cumbersome. The details of the computation will be worked out in the case of isotropic elasticity.

5. The isotropic case. The equations (5) can be written in the isotropic case

$$\{(b^2\, \partial^2/\partial x_k^2 - \partial^2/\partial t^2)\, \delta_{ij} + (a^2 - b^2)\, \partial^2/\partial x_i\, \partial x_j\} u_j = -f_i \,. \tag{26}$$

The operators D_{ij} are simplified to

$$D_{ij} = (a^2\, \Delta - \partial^2/\partial t^2)\, \delta_{ij} - (a^2 - b^2)\, \partial^2/\partial x_i\, \partial x_j \,, \tag{27}$$

and the relation (9) becomes

$$\{(b^2\, \Delta - \partial^2/\partial t^2)\, \delta_{ij} + (a^2 - b^2)\, \partial^2/\partial x_i\, \partial x_j\}\, D_{jk} = (a^2\, \Delta - \partial^2/\partial t^2)(b^2\, \Delta - \partial^2/\partial t^2)\, \delta_{ik} \,, \tag{28}$$

where Δ is the Laplacian. The kernel K of (10) is now

$$K(t, x, y, z) = (2\pi)^{-3} \iiint d^3k (b^{-1} \sin bkt - a^{-1} \sin akt)(a^2 - b^2)^{-1} k^{-3} \exp (i\mathbf{k}\cdot\mathbf{x}) \tag{29}$$

with $k^2 = k_x^2 + k_y^2 + k_z^2$. K and its first two derivatives with respect to time vanish at $t = 0$, whereas its third derivative is $\delta(x)\delta(y)\delta(z)$ at $t = 0$. The integration can be performed explicitly

$$K = \frac{1}{4\pi} \begin{cases} 0 & \text{for} \quad at < r, \\ (a^2 - b^2)^{-1}(t/r - 1/a) & \text{for} \quad bt < r < at, \\ [ab(a + b)]^{-1} & \text{for} \quad r < bt. \end{cases} \tag{30}$$

With an arbitrary source distribution $f_i(t', x', y', z')$ and with K written as function of $t - t'$, $x - x'$, etc. one has

$$\int_{-\infty}^{t} dt' \iiint d^3x' K f_i = (4\pi)^{-1} \iiint d^3x' \left\{ \int_{-\infty}^{t-r/b} dt' \frac{f_i}{ab(a + b)} + \int_{t-r/b}^{t-r/a} dt' \left(\frac{t - t'}{r} - \frac{1}{a} \right) f_i \right\}. \tag{31}$$

The operator D_{ij} acting on this expression leads to surface terms because of the peculiar limits on the t-integration. The i-component of the displacement due to the body forces f_j is then given by

$$(4\pi)^{-1} \iiint d^3x' \left\{ b^{-2}r^{-1}f_i(t - r/b) + \partial^2(1/r)/\partial x_i \, \partial x_j \int_{r/a}^{r/b} dt' \cdot t' f_j(t - t') \right.$$
$$\left. + r^{-2}(x_i - x_i')(x_j - x_j')[a^{-2}r^{-1}f_j(t - r/a) - b^{-2}r^{-1}f_j(t - r/b)] \right\}, \quad (32)$$

where the time variable in f_i is explicitly indicated, the spatial arguments of f_i are x', y', z', and r is equal to $((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}$. This last formula is essentially the same as the one obtained by Love [3].

In the isotropic medium the coefficients (6) are

$$\gamma_{ij,kl} = b^2 \, \delta_{ij} \, \delta_{kl} + (a^2 - b^2) \, \delta_{ik} \, \delta_{jl} \quad (33)$$

as can be seen directly from (5) and (26). The Burgers vector β_i has now the components $(\beta, 0, 0)$, since it is natural for the Burgers vector to lie in the slip plane $z = 0$, and the x-coordinate may then be chosen parallel to the Burgers vector. The differentiation in (25) is simply $\beta b^2 \delta_{1j} \partial/\partial z$. The integral in (25) corresponds exactly to the integral (32) with $f_i = \beta b^2 \delta_{1i}$ concentrated in the space $z = 0$ inside the curves C_t. Therefore the displacements v_k are given by the formula

$$(4\pi)^{-1}\beta \, \partial/\partial z \cdot \iint dx' \, dy' \left\{ \frac{\delta_{1k}}{r} \bigg|_{t'=t-r/b} + \frac{\partial^2(1/r)}{\partial x_1 \, \partial x_k} b^2 \int_{t-r/b}^{t-r/a} dt'(t - t') \right.$$
$$\left. + \frac{(x_1 - x_1')(x_k - x_k')}{r^2} \left(\frac{b^2}{a^2 r} \bigg|_{t'=t-r/a} - \frac{1}{r} \bigg|_{t'=t-r/b} \right) \right\}, \quad (34)$$

where the integration is extended over the inferior of C_t.

It is not immediately apparent that this expression is discontinuous as z approaches 0. To show this property let us consider the first term which occurs only for $k = 1$, i.e. only for the first component of the displacement,

$$-(4\pi)^{-1}\beta \iint r^{-3}z \, dx' \, dy', \quad (35)$$

where the integration is extended over the interior of C_t. This is interpreted as a surface integral in the (t', x', y') space, over the wave surface

$$b(t - t') = ((x - x')^2 + (y - y')^2 + z^2)^{1/2} = r, \quad (36)$$

as far as it lies inside the curves C_t. As z approaches 0, the main contribution to the integral comes from the point (x', y') in the immediate neighborhood of the values (x, y), provided the point (x, y) lies inside the curves C_t at time t. Moreover since the integral (35) over (x', y') converges without restriction to the interior of the curves C_t, the contribution of this boundary to the integral (35) vanishes as z approaches 0. Therefore (35) becomes in the limit of vanishing z

$$-(4\pi)^{-1}\beta \lim_{|z|\to 0} \iint \frac{z \, dx' \, dy'}{[(x - x')^2 + (y - y')^2 + z^2]^{3/2}} \quad (37)$$

integrated over all (x', y') space. This last integral is $-\beta/2$ sign z, and it has just the discontinuity at $z = 0$ which is required by (22). By the same method the integral (35) is seen to vanish for the points in the z-plane outside the curves C_t. The various parts of (34) can be investigated by similar arguments.

References

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