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BOUNDARY CONTRACTION METHOD FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS: CONVERGENCE AND BOUNDARY CONDITIONS*

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1. Introduction. In previous papers by Milnes and Potts [1, 2] and Chow and Milnes [3] a method for numerical solution of partial differential equations has been introduced and applied specifically to the Dirichlet problem for the unit circle. In a later paper [4], Chow and Milnes applied the boundary contraction technique to the solution of the Laplace's differential equation with boundary conditions of the Neumann and mixed type. In addition, the Dirichlet problem was solved by developing a stable star from the difference approximation to the Laplace's equation without utilizing the analytic solution of the Laplace equation as given by the Poisson's integral. The boundary contraction technique as based on the difference approximation to partial differential equations was further applied by Chow and Milnes to a class of partial differential equations of the hyperbolic-parabolic type [5]. There the stability of the difference star was proved by examining the magnitude of the eigenvalues of the solution of the difference equation and an error bound due to truncation is also given.

The purpose of this paper is to give a theoretical discussion of the boundary contraction method as applied to a general partial differential equation which includes the Laplace's differential equation and the bi-harmonic equation as special cases. It is divided into two parts.

In Part I we discuss the problem of convergence of the boundary contraction method as the mesh size tends to zero. The divided difference equations which approximate the given differential equations are first solved analytically (Sections 2 and 3), and the solution is examined in the limit as the mesh size tends to zero (Section 4). It is proved that in the limit the approximate solution of the difference equations approaches the Fourier series expansion solution of the given partial differential equations.

In Part II we investigate, with special reference to the boundary contraction method, the manner in which the attached boundary conditions may be prescribed, so that the problem is properly posed; that is to say, the conditions are such that they imply the existence of a unique and bounded solution for the problem. It is made apparent (Section 5) that the boundary conditions may not be prescribed in a haphazard fashion but must satisfy very definite relations among themselves. These relationships are developed

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(Sections 6 and 7), and are presented in full detail with the principal conclusions presented in section 8. It is shown that in order to achieve a stable star the boundary conditions must be prescribed in the right manner so that the problem is neither over-determined nor under-determined.

Part I. Convergence

2. Difference approximations. We consider the following linear partial differential equation of order p and homogeneous in r:

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} \frac{a_{\alpha\beta}}{r^{p-\alpha+\beta}} \frac{\partial^\alpha u}{\partial r^{\alpha-\beta}\,\partial\theta^\beta} = 0, \tag{2.1}$$

where the coefficient $a_{\alpha\beta}$ are real and independent of the radius r and the polar angle θ. The region of interest is a circle with center at $r = 0$, and since (2.1) is homogeneous in r, there is no loss of generality by restricting this region to the unit circle $r = 1$. Let the values of $u(r, \theta)$, together with its normal derivatives

$$\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \cdots \frac{\partial^i u}{\partial r^i}$$

be properly specified on the boundary of the circle $r = 1$; it is required to find the unique and bounded solution $u(r, \theta)$ for $r < 1$.

Before we replace Eq. (2.1) by finite difference approximations and solve the difference equation by the method of boundary contraction, it is convenient to introduce a polar grid system consisting of a series of concentric circles C_0, C_1, C_2, $\cdots$ and N equally spaced radial lines, so that the angle between each two consecutive radial lines is $2\pi/N$. The concentric circles are not equally spaced, so that if the respective radii are r_0, r_1, r_2, $\cdots$ we have

$$r_{i+1} = \rho r_i, \qquad (i = 0, 1, 2, \cdots), \tag{2.2}$$

where ρ is a constant and $r_0 = 1$. The values of $u(r, \theta)$ at $r = r_i$, $\theta = \theta_j$, will be denoted, for simplicity, by $u(i, j)$, with the second index reduced modulo N, and increasing in the counterclockwise sense, Fig. 1. Furthermore, let

$$\begin{aligned}
&(\lambda - 1)(\lambda - \rho) \cdots (\lambda - \rho^{\alpha-1}) = \lambda^\alpha + s_{\alpha,1}\lambda^{\alpha-1} + s_{\alpha,2}\lambda^{\alpha-2} + \cdots + s_{\alpha,\alpha}, \\
&\sigma_\alpha = 1 + 2 + 3 + \cdots + (\alpha - 1) = \frac{\alpha(\alpha-1)}{2}, \\
&\psi u(m, n) = u(m, n + 1), \\
&\Delta_\alpha u(m, n) = u(m, n) + s_{\alpha,1}u(m - 1, n) + \cdots + s_{\alpha,\alpha}u(m - \alpha, n).
\end{aligned} \tag{2.3}$$

Replacing the derivatives in (2.1) by divided difference approximations, we have

$$\frac{\partial^\alpha}{\partial r^{\alpha-\beta}\,\partial\theta^\beta} u(i, j) \approx \frac{1}{r^{\alpha-\beta}\,\Delta\theta^\beta} \cdot \frac{(-)^{\alpha-\beta}(-)^\beta(\alpha-\beta)!}{\rho^{\sigma_{\alpha-\beta}}(1-\rho)(1-\rho^2)\cdots(1-\rho^{\alpha-\beta})} \Delta_{\alpha-\beta}(1 - \psi)^\beta u\left(i + \alpha - \beta, j - \frac{\beta}{2}\right) \tag{2.4A}$$

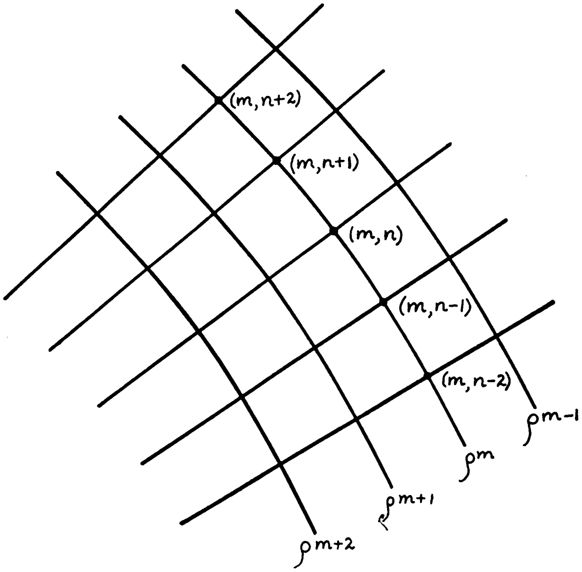

FIG. 1.

if β is even, and

$$\approx \frac{1}{r^{\alpha-\beta}\,\Delta\theta^{\beta}} \frac{(-)^{\alpha-\beta}(-)^{\beta}(\alpha-\beta)!}{\rho^{\sigma\alpha-\beta}(1-\rho)(1-\rho^2)\cdots(1-\rho^{\alpha-\beta})}$$

$$\cdot \Delta_{\alpha-\beta}(1-\psi)^{\beta}u\left(i+\alpha-\beta,\, j-\frac{\beta+1}{2}\right) \qquad (2.4\text{B})$$

if β is odd, ($i = 0, 1, 2, \cdots ; j = 0, 1, 2, \cdots , (N-1)$). Then, substitution of (2.4) into (2.1) yields

$$\sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha} A_{\alpha\beta}\,\Delta_{\alpha-\beta}(1-\psi)^{\beta}\left\{u\left(i+\alpha-\beta,\, j-\frac{\beta}{2}\right),\, u\left(i+\alpha-\beta,\, j-\frac{\beta+1}{2}\right)\right\} = 0, \qquad (2.5)$$

where the first or second member in the parenthesis is to be taken depending on whether β is even or odd, and

$$A_{\alpha\beta} = \frac{(-)^{\alpha}(\alpha-\beta)!}{\rho^{\sigma\alpha-\beta}(1-\rho)(1-\rho^2)\cdots(1-\rho^{\alpha-\beta})}\frac{a_{\alpha\beta}}{\Delta\theta^{\beta}}. \qquad (2.6)$$

Now, if β is even,

$$A_{\alpha\beta}\,\Delta_{\alpha-\beta}(1-\psi)^{\beta}u\left(i+\alpha-\beta,\, j-\frac{\beta}{2}\right)$$

$$= A_{\alpha\beta}\left\{\left[u\left(i+\alpha-\beta,\, j-\frac{\beta}{2}\right) + s_{\alpha-\beta,1}u\left(i+\alpha-\beta-1,\, j-\frac{\beta}{2}\right)\right.\right.$$

$$\left. + \cdots + s_{\alpha-\beta,\alpha-\beta}u\left(i,\, j-\frac{\beta}{2}\right)\right] - \binom{\beta}{1}\left[u\left(i+\alpha-\beta,\, j-\frac{\beta}{2}+1\right)\right.$$

$$\left. + s_{\alpha-\beta,1}u\left(i+\alpha-\beta-1,\, j-\frac{\beta}{2}+1\right) + \cdots + s_{\alpha-\beta,\alpha-\beta}u\left(i,\, j-\frac{\beta}{2}+1\right)\right] + \cdots$$

$$+ (-)^{\beta/2} \begin{Bmatrix} \beta \\ \frac{\beta}{2} \end{Bmatrix} [u(i + \alpha - \beta, j) + s_{\alpha-\beta,1} u(i + \alpha - \beta - 1, j)$$

$$+ \cdots + s_{\alpha-\beta,\alpha-\beta} u(i, j)] + \cdots + \binom{\beta}{\beta}\left[u\left(i + \alpha - \beta, j + \frac{\beta}{2}\right)\right.$$

$$\left.\left. + s_{\alpha-\beta,1} u\left(i + \alpha - \beta - 1, j + \frac{\beta}{2}\right) + \cdots + s_{\alpha-\beta,\alpha-\beta} u\left(i, j + \frac{\beta}{2}\right)\right]\right\}$$

$$= A_{\alpha\beta}\left[1, -\binom{\beta}{1}, \cdots, (-)^{\beta/2} \begin{Bmatrix} \beta \\ \frac{\beta}{2} \end{Bmatrix}, \cdots, \binom{\beta}{\beta}\right]$$

$$\cdot\left[\left\{u\left(i + \alpha - \beta, j - \frac{\beta}{2}\right), u\left(i + \alpha - \beta, j - \frac{\beta}{2} + 1\right), \cdots, \right.\right.$$

$$\left. u(i + \alpha - \beta, j), \cdots, u\left(i + \alpha - \beta, j + \frac{\beta}{2}\right)\right\}^*$$

$$+ s_{\alpha-\beta,1}\left\{u\left(i + \alpha - \beta - 1, j - \frac{\beta}{2}\right), u\left(i + \alpha - \beta - 1, j - \frac{\beta}{2} + 1\right), \cdots, \right.$$

$$\left. u(i + \alpha - \beta - 1, j), \cdots, u\left(i + \alpha - \beta - 1, j + \frac{\beta}{2}\right)\right\}^* + \cdots$$

$$\left. + s_{\alpha-\beta,\alpha-\beta}\left\{u\left(i, j - \frac{\beta}{2}\right), u\left(i, j - \frac{\beta}{2} + 1\right), \cdots, u(i, j), \cdots, u\left(i, j + \frac{\beta}{2}\right)\right\}^*\right],$$

$$(i = 0, 1, 2, \cdots; j = 0, 1, 2, \cdots, (N - 1)), \tag{2.7}$$

where $\{\cdot, \cdots\cdots, \cdot\cdot\}^*$ denotes a column vector. Furthermore, if we define

$$\mathbf{U}_m = \left\{u\left(m, -\frac{p}{2}\right), u\left(m, -\frac{p}{2} + 1\right), \cdots, \right.$$

$$\left. u(m, N), u(m, 1), \cdots, u\left(m, -\frac{p}{2} - 1\right)\right\}^* \tag{2.8A}$$

when p is even and

$$U_m = \left\{u\left(m, -\frac{p+1}{2}\right), u\left(m, -\frac{p-1}{2}\right), \cdots, \right.$$

$$\left. u(m, N), u(m, 1), \cdots u\left(m, -\frac{p+3}{2}\right)\right\}^* \tag{2.8B}$$

when p is odd, we can rewrite (2.7) as

$$A_{\alpha\beta}\, \Delta_{\alpha\beta}(1 - \psi)^\beta u\left(i + \alpha - \beta, j - \frac{\beta}{2}\right)$$

$$= A_{\alpha\beta}\left[0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta/2} \begin{Bmatrix} \beta \\ \frac{\beta}{2} \end{Bmatrix}, \cdots, \binom{\beta}{\beta}, 0, \cdots 0\right] \cdot \tag{2.9}$$

$$\cdot [\mathbf{U}_{i+\alpha-\beta} + s_{\alpha-\beta,1} \mathbf{U}_{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta} \mathbf{U}_i],$$

$$(i = 0, 1, 2, \cdots; j = 0, 1, 2, \cdots, (N - 1)),$$

where the row vector $[0, \cdots 0, 1, -(\beta/1), \cdots, (\beta/\beta), 0, \cdots.]$ is such that there are $j + \frac{1}{2}(p - \beta)$ or $j + \frac{1}{2}(p - \beta - 1)$ zero elements preceding the element 1, depending on whether p is even or odd. If β is odd, we similarly obtain the following expression:

$$A_{\alpha\beta}\, \Delta_{\alpha-\beta}(1 - \psi)^{\beta} u\left(i + \alpha - \beta, j - \frac{\beta + 1}{2}\right)$$
$$= A_{\alpha\beta}\left[0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots 0\right]$$
$$\cdot[\mathbf{U}_{i+\alpha-\beta} + s_{\alpha-\beta,1}\mathbf{U}_{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta}\mathbf{U}_i],$$
$$(i = 0, 1, 2, \cdots; j = 0, 1, 2, \cdots, (N - 1)), \qquad (2.10)$$

where the row vector

$$\left[0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots 0\right]$$

is such that there are $j + (p - \beta - 1)/2$ or $j + (p - \beta)/2$ zero elements preceding the element 1 depending on whether p is even or odd.

With the results (2.9) and (2.10) it is possible to write the difference approximation to Eq. (2.1) as follows

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} A_{\alpha\beta}\left[0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots 0\right]$$
$$\cdot[\mathbf{U}_{i+\alpha-\beta} + s_{\alpha-\beta,1}\mathbf{U}_{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta}\mathbf{U}_i] = 0,$$
$$(i = 0, 1, 2, \cdots; j = 0, 1, 2, \cdots, (N - 1)), \qquad (2.11)$$

where the row vector

$$\left[0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots 0\right]$$

is such that the number of zero elements preceding the element 1 will vary with j, p and β as already described. Let j range over its possible set of values in (2.9) and (2.10), i.e., let us apply the difference equation to each successive nodal point in order on a circle, then we arrive at $(N - 1)$ additional equations similar to (2.11). Due to the cyclic property of the second index j of $u(i, j)$, these equations can be compactly written in the following matrix form:

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} A_{\alpha\beta} C_{\beta}\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots, 0\right)$$
$$\cdot[\mathbf{U}_{i+\alpha-\beta} + s_{\alpha-\beta,1}\mathbf{U}_{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta}\mathbf{U}_i] = 0, \qquad (2.12)$$

where

$$C_{\beta}\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots, 0\right)$$

is an $N \times N$ circulant matrix of which the elements of the first row are

$$0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^{\beta}\binom{\beta}{\beta}, 0, \cdots, 0$$

with $\frac{1}{2}(p - \beta)$ or $\frac{1}{2}(p - \beta - 1)$ zeros preceding 1, according as p is even or odd when β is even, and $\frac{1}{2}(p - \beta + 1)$ or $\frac{1}{2}(p - \beta)$ zeros preceding 1, according as p is even or odd when β is odd.

$$C_\beta\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right)$$

$$= \begin{bmatrix}
0 & \cdots & 0 & 1 & -\binom{\beta}{1} & \cdots & (-)^\beta\binom{\beta}{\beta} & 0 & \cdots & 0 \\
\vdots & & & \ddots & \ddots & & & \ddots & \ddots & \vdots \\
\vdots & & & & \ddots & \ddots & & & \ddots & 0 \\
0 & & & & & \ddots & \ddots & & & (-)^\beta\binom{\beta}{\beta} \\
(-)^\beta\binom{\beta}{\beta} & \ddots & & & & & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & & & & \ddots & \ddots & 1 \\
1 & & \ddots & \ddots & & & & & \ddots & 0 \\
0 & \ddots & & \ddots & \ddots & & & & & \vdots \\
\vdots & \ddots & \ddots & & \ddots & \ddots & & & & \vdots \\
0 & \cdots & 0 & 1 & -\binom{\beta}{1} & \cdots & (-)^\beta\binom{\beta}{\beta} & 0 & \cdots & 0
\end{bmatrix}. \tag{2.13}$$

3. Solution of the difference equations. We now attempt to obtain the general solution of the approximate difference equations in matrix form (2.12), without recourse to boundary conditions. Since the stability of computation will depend on the boundedness of the solution as $r \rightarrow 0$, the question of how to specify the boundary conditions properly in order to ensure a bounded solution will be left for further consideration in Part II of this paper.

Let us try a solution of the following form assuming $\mathbf{U}_0 \neq 0$:

$$\mathbf{U}_i = X^i\mathbf{U}_0, \qquad (i = 0, 1, 2, \cdots), \tag{3.1}$$

where X is an $N \times N$ circulant. Substitution of (3.1) into Equation (2.12) results in the following expression:

$$\sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha} A_{\alpha\beta}C_\beta\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right) \cdot[X^{i+\alpha-\beta} + s_{\alpha-\beta,1}X^{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta}X^i]\mathbf{U}_0 = 0. \tag{3.2}$$

Now consider the following complete ortho-normal set of vectors $\chi_0, \chi_1, \cdots, \chi_{N-1}$ in N dimensional space:

$$\chi_i = N^{-1/2}\{1, \omega^i, \omega^{2i}, \cdots, \omega^{(N-1)i}\}^*, \qquad (i = 0, 1, 2, \cdots, (N-1)), \tag{3.3}$$

$$\omega = \exp\left(\frac{2\pi\iota}{N}\right), \qquad \iota = (-1)^{1/2}, \tag{3.4}$$

and resolve $\mathbf{U}_0$ with respect to this basis:

$$\mathbf{U}_0 = \sum_{j=0}^{N-1} \mu_{0j}\chi_j . \tag{3.5}$$

Substitution of (3.5) into (3.2) gives

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} \sum_{j=0}^{N-1} A_{\alpha\beta}\mu_{0j}C_\beta\left(0, \cdots 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right)$$
$$\cdot[X^{i+\alpha-\beta} + s_{\alpha-\beta,1}X^{i+\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta}X^i]\chi_j = 0. \tag{3.6}$$

It is known that an $N \times N$ circulant matrix $C(a_0, a_1, \cdots, a_{N-1})$ has N eigenvectors given by (3.3) which are independent of the elements of the matrix. The corresponding eigenvalues are, [6]:

$$\lambda_j = \sum_{k=0}^{N-1} a_k\omega^{jk}, \qquad (j = 0, 1, 2, \cdots, (N-1)). \tag{3.7}$$

We denote the N eigenvalues of X by $\xi_0, \xi_1, \cdots, \xi_{N-1}$ corresponding to $\chi_0, \chi_1, \cdots, \chi_{N-1}$. Following the details of the circulant

$$C_\beta\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right)$$

already given in Section 2, we have if β is even

$$C_\beta\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right)\chi_j$$
$$\begin{aligned} &= \omega^{j(p-\beta)/2}(1-\omega^j)^\beta\chi_j \quad \text{when } p \text{ is even,}\\ &= \omega^{j(p-\beta-1)/2}(1-\omega^j)^\beta\chi_j \quad \text{when } p \text{ is odd,}\end{aligned} \tag{3.8}$$

and if β is odd

$$C_\beta\left(0, \cdots, 0, 1, -\binom{\beta}{1}, \cdots, (-)^\beta\binom{\beta}{\beta}, 0, \cdots, 0\right)\chi_j$$
$$\begin{aligned} &= \omega^{j(p-\beta+1)/2}(1-\omega^j)^\beta\chi_j \quad \text{when } p \text{ is even,}\\ &= \omega^{j(p-\beta)/2}(1-\omega^j)^\beta\chi_j \quad \text{when } p \text{ is odd.}\end{aligned} \tag{3.9}$$

Using these results, we reduce the matrix equation (3.6) to an algebraic equation in ξ_j when p is even

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} \sum_{j=0}^{N-1} A_{\alpha\beta}\mu_{0j}(\omega^{j(p-\beta)/2}, \omega^{j(p-\beta+1)/2})(1-\omega^j)^\beta$$
$$\cdot(\xi_j^{\alpha-\beta} + s_{\alpha-\beta,1}\xi_j^{\alpha-\beta-1} + \cdots + s_{\alpha-\beta,\alpha-\beta})\chi_j = 0, \tag{3.10}$$

and it follows immediately from (3.10) by making use of (2.3) that

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} A_{\alpha\beta}(\omega^{j(p-\beta)/2}, \omega^{j(p-\beta+1)/2})(1-\omega^j)^\beta(\xi_j - 1)(\xi_j - \rho) \cdots (\xi_j - \rho^{\alpha-\beta-1}) = 0,$$
$$(j = 0, 1, 2, \cdots, (N-1)), \tag{3.11}$$

where the first or second member in the parenthesis is to be taken according as β is even or odd. Clearly this expression is also true for odd p. This equation is of p-th degree in ξ_j : let the p roots be

$$\xi_{j1}, \xi_{j2}, \cdots, \xi_{jp}, \qquad (j = 0, 1, 2, \cdots, (N-1)). \tag{3.12}$$

Corresponding to the p eigenvalues we have p circulants

$$X_1, X_2, \cdots, X_p \tag{3.13}$$

such that

$$X_k = Q \operatorname{diag}(\xi_{0k}, \xi_{1k}, \cdots, \xi_{N-1,k})\bar{Q}, \qquad (k = 1, 2, \cdots, p), \tag{3.14}$$

and

$$Q = \frac{1}{N^{1/2}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix}. \tag{3.15}$$

The general solution of the difference equations in matrix form (2.12) can therefore be expressed as

$$\mathbf{U}_i = (\mathcal{E}_1 X_1^i + \mathcal{E}_2 X_2^i + \cdots + \mathcal{E}_p X_p^i)\mathbf{U}_0, \qquad (i = 0, 1, 2, \cdots), \tag{3.16}$$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p$ are circulants involving arbitrary constants which are to be determined by given boundary data:

$$\mathcal{E}_k = C(e_{k1}, e_{k2}, \cdots, e_{k,N-1}), \qquad (k = 1, 2, \cdots, p). \tag{3.17}$$

If we put $i = 0$ in (3.16) we have

$$\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_p = I, \tag{3.18}$$

I being the identity matrix. Thus we see $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p$ are not entirely independent, the number of arbitrary constants being $N(p-1)$. On the other hand, since $\mathbf{U}_0$ is arbitrary, the combined number of arbitrary constants in the solution as given by (3.16) is Np.

Since circulant matrices commute in multiplication, it is easy to see that the solution given by Equation (3.16) satisfies Equation (2.12); moreover, it involves Np arbitrary constants, so it is the general solution.

4. Comparison with Fourier series expansion solution. At this point it is appropriate to investigate the solution found in the previous paragraph as given by (3.16) in more detail; in particular, it is a matter of interest to examine how the solution of the difference equation passes over to the analytic solution of the differential equation in the limit as the mesh size tends to zero.

We first resolve $\mathbf{U}_0$ with respect to the basis $\chi_0, \chi_1, \cdots, \chi_{N-1}$ as given by (3.5). The coefficients of the expansion can be found by taking the inner product, thus

$$\mu_{0j} = (\mathbf{U}_0, \bar{\chi}_j), \tag{4.1}$$

where $\bar{\chi}_j$ is the complex conjugate of $\chi_j (j = 0, 1, 2, \cdots, (N-1))$. Substituting Equation (3.5) into Equation (3.16), we get

$$\mathbf{U}_i = \sum_{j=0}^{N-1} \mu_{0j}(\mathcal{E}_1 X_1^i + \mathcal{E}_2 X_2^i + \cdots + \mathcal{E}_p X_p^i)\chi_j$$
$$= \sum_{j=0}^{N-1} \mu_{0j}(\lambda_{j1}\xi_{j1}^i + \lambda_{j2}\xi_{j2}^i + \cdots + \lambda_{jp}\xi_{jp}^i)\chi_j, \qquad (i = 0, 1, 2, \cdots), \qquad (4.2)$$

where λ_{jk} is an eigenvalue of $\mathcal{E}_k$, $(j = 0, 1, 2, \cdots, (N-1))$. If the components of the vector $\mathbf{U}_i$ are $u_{i0}, u_{i1}, \cdots, u_{i,N-1}$, then the $(s+1)$th component of the above equation is

$$u_{is} = N^{-1/2} \sum_{j=0}^{N-1} \mu_{0j}(\lambda_{j1}\xi_{j1}^i + \lambda_{j2}\xi_{j2}^i + \cdots + \lambda_{jp}\xi_{jp}^i)\omega^{sj},$$
$$(i = 0, 1, 2, \cdots; s = 0, 1, 2, \cdots, (N-1)). \qquad (4.3)$$

If N is odd, then,

$$u_{is} = N^{-1/2}\left(\sum_{j=0}^{(N-1)/2} + \sum_{j=(N+1)/2}^{N-1}\right)\mu_{0j}(\lambda_{j1}\xi_{j1}^i + \lambda_{j2}\xi_{j2}^i + \cdots + \lambda_{jp}\xi_{jp})\omega^{sj}$$
$$= N^{-1/2}\mu_{00}(\lambda_{01}\xi_{01}^i + \lambda_{02}\xi_{02}^i + \cdots + \lambda_{0p}\xi_{0p})$$
$$+ N^{-1/2} \sum_{j=1}^{(N-1)/2} \mu_{0j}(\lambda_{j1}\xi_{j1}^i + \lambda_{j2}\xi_{j2}^i + \cdots + \lambda_{jp}\xi_{jp}^i)\omega^{sj}$$
$$+ N^{-1/2} \sum_{j=1}^{(N-1)/2} \mu_{0,N-j}(\lambda_{N-j,1}\xi_{N-j,1}^i + \lambda_{N-j,2}\xi_{N-j,2}^i + \cdots + \lambda_{N-j,p}\xi_{N-j,p}^i)\omega^{-sj},$$
$$(i = 0, 1, 2, \cdots; s = 0, 1, 2, \cdots, (N-1)). \qquad (4.4)$$

Since $\omega = \exp(2\pi\iota/N)$, it follows, if we write $\theta_s = 2\pi s/N$,

$$u_{is} = c_{01}\xi_{01}^i + c_{02}\xi_{02}^i + \cdots + c_{0p}\xi_{0p}^i$$
$$+ \sum_{j=1}^{(N-1)/2} (c_{j1}\xi_{j1}^i + c_{j2}\xi_{j2}^i + \cdots + c_{jp}\xi_{jp}^i) \exp(j\theta_s\iota)$$
$$+ \sum_{j=1}^{(N-1)/2} (c_{N-j,1}\xi_{N-j,1}^i + c_{N-j,2}\xi_{N-j,2}^i + \cdots + c_{N-j,p}\xi_{N-j,p}^i) \exp(-j\theta_s\iota), \qquad (4.5)$$

where

$$c_{jk} = N^{-1/2}(\mathbf{U}_0, \bar{\chi}_j)\lambda_{jk},$$
$$c_{N-j,k} = N^{-1/2}(\mathbf{U}_0, \bar{\chi}_{N-j})\lambda_{N-j,k}$$
$$= N^{-1/2}(\mathbf{U}_0, \chi_j)\lambda_{N-j,k}, \qquad (j = 1, 2, \cdots, \tfrac{1}{2}(N-1); k = 1, 2, \cdots, p). \quad (4.6)$$

Due to the circulant nature of the real matrices $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p$, we have $\lambda_{jk} = \bar{\lambda}_{N-j,k}$. Thus if $\mathbf{U}_0$ is real, then

$$c_{jk} = \bar{c}_{N-j,k}, \qquad (j = 1, 2, \cdots, \tfrac{1}{2}(N-1); k = 1, 2, \cdots, p). \qquad (4.7)$$

If N is even, an expression similar to (4.5) exists. Now in Equation (4.5), take only the first n harmonics such that $n \ll \frac{1}{2}(N - 1)$:

$$\begin{aligned} u_{is} = {} & c_{01}\xi_{01}^{i} + c_{02}\xi_{02}^{i} + \cdots + c_{0p}\xi_{0p}^{i} \\ & + \sum_{j=1}^{n} (c_{j1}\xi_{j1}^{i} + c_{j2}\xi_{j2}^{i} + \cdots + c_{jp}\xi_{jp}^{i}) \exp (j\theta_s\iota) \\ & + \sum_{j=1}^{n} (c_{N-j,1}\xi_{N-j,1}^{i} + c_{N-j,2}\xi_{N-j,2}^{i} + \cdots + c_{N-j,p}\xi_{N-j,p}^{i}) \exp (-j\theta_s\iota) \\ & + R_n, \end{aligned} \tag{4.8}$$

so that

$$\begin{aligned} R_n = {} & \sum_{j=n+1}^{(N-1)/2} (c_{j1}\xi_{j1}^{i} + c_{j2}\xi_{j2}^{i} + \cdots + c_{jp}\xi_{jp}^{i}) \exp (j\theta_s\iota) \\ & + \sum_{j=n+1}^{(N-1)/2} (c_{N-j,1}\xi_{N-j,1}^{i} + c_{N-j,2}\xi_{N-j,2}^{i} + \cdots + c_{N-j,p}\xi_{N-j,p}^{i}) \exp (-j\theta_s\iota). \end{aligned} \tag{4.9}$$

With c_{jk}, $c_{N-j,k}$ given by (4.6) we can write

$$c_{jk} = \frac{\lambda_{jk}}{N} \sum_{l=0}^{N-1} u_{0l} \exp \left(-\frac{2\pi jl\iota}{N}\right). \tag{4.10}$$

Using Abel's transformation [7] we rewrite (4.10) as

$$\begin{aligned} c_{jk} &= \frac{\lambda_{jk}}{N} \left\{ \sum_{k=0}^{N-2} [u_{0k} - u_{0,k+1}] \sum_{l=0}^{k} \exp \left(-\frac{2\pi jl\iota}{N}\right) + \sum_{l=0}^{N-1} u_{0,N-1} \exp \left(-\frac{2\pi jl\iota}{N}\right) \right\} \\ &= \frac{\lambda_{jk}}{N} \sum_{k=0}^{N-1} [u_{0k} - u_{0,k+1}] \frac{1 - \exp [-2\pi j(k+1)\iota/N]}{1 - \exp [-2\pi j\iota/N]}, \qquad (j \neq 0). \end{aligned}$$

after making use of the fact that $\sum_{l=0}^{N-1} \exp (-2\pi jl\iota/N) = 0$. Thus

$$c_{jk} = \frac{\lambda_{jk}}{N^2} \frac{2\pi}{\exp (2\pi j\iota/N) - 1} \sum_{k=0}^{N-1} \frac{N}{2\pi} [u_{0,k+1} - u_{0k}] \exp \left(-\frac{2\pi jk\iota}{N}\right). \tag{4.11}$$

If we define

$$u_{0l}^{(1)} = \frac{N}{2\pi} [u_{0,l+1} - u_{0l}], \tag{4.12}$$

$$\lambda_{jk}^{(1)} = \frac{2\pi\lambda_{jk}}{N[\exp (2\pi j\iota/N) - 1]}, \tag{4.13}$$

we then have

$$c_{jk} = \frac{\lambda_{jk}^{(1)}}{N} \sum_{l=0}^{N-1} u_{0l}^{(1)} \exp \left(-\frac{2\pi jl\iota}{N}\right), \tag{4.14}$$

which is similar to (4.10). Repeating the process ν times gives

$$c_{jk} = \frac{\lambda_{jk}^{(\nu)}}{N} \sum_{l=0}^{N-1} u_{0l}^{(\nu)} \exp \left(-\frac{2\pi jl\iota}{N}\right), \tag{4.15}$$

where

$$\lambda_{jk}^{(\nu)} = \left\{\frac{2\pi}{N[\exp(2\pi j\iota/N) - 1]}\right\}^{\nu} \lambda_{jk}, \tag{4.16}$$

and $u_{0l}^{(\nu)}$ corresponds to the forward difference approximation to the νth derivative of $u(1, \theta)$ with respect to θ for $\theta = 2\pi l/N$. Now as N approaches infinity it follows from (4.16) that $\lambda_{jk}^{(\nu)}$ approaches $\lambda_{jk}/j^{\nu}\iota^{\nu}$ and $u_{0l}^{(\nu)}$ will approach $\partial^{\nu}u(r, \theta)/\partial\theta^{\nu}$ for $r = 1, \theta = 2\pi l/N$. Assuming the boundedness of $\partial^{\nu}u(r, \theta)/\partial\theta^{\nu}$ we obtain the following inequality:

$$|c_{jk}| \leqq \frac{|\lambda_{jk}|}{j^{\nu}} \max \left|\frac{\partial^{\nu}}{\partial\theta^{\nu}} u(r, \theta)\right|_{r=1}, \quad \begin{bmatrix} j = 0, 1, 2, \cdots \\ k = 1, 2, 3, \cdots, p \end{bmatrix} \tag{4.17}$$

Next we show that $|\lambda_{jk}|$, $(j = 0, 1, 2, \cdots; k = 1, 2, 3, \cdots, p)$ remain bounded as $N \to \infty$ and $\rho \to 1$. From Equation (4.3)

$$u_{is} = N^{-1/2} \sum_{j=0}^{N-1} \mu_{0j}(\lambda_{j1}\xi_{j1}^{i} + \lambda_{j2}\xi_{j2}^{i} + \cdots + \lambda_{jp}\xi_{jp}^{i})\omega^{sj},$$

$$(i = 0, 1, 2, \cdots; s = 0, 1, 2, \cdots, (N - 1)). \tag{4.18}$$

With $s = 0, 1, 2, \cdots, (N - 1)$ we multiply the resulting equations taken in order by $1, \omega^{-m}, \omega^{-2m}, \cdots, \omega^{-(N-1)m}$, $(m = 0. 1, 2, \cdots, (N - 1))$ and add. This results

$$\sum_{s=0}^{N-1} u_{is}\omega^{-ms} = N^{1/2}\mu_{0m}(\lambda_{m1}\xi_{m1}^{i} + \lambda_{m2}\xi_{m2}^{i} + \cdots + \lambda_{mp}\xi_{mp}^{i}),$$

$$(m = 0, 1, 2, \cdots, (N - 1)). \tag{4.19}$$

And if we replace i in (4.19) by $0, i, 2i, \cdots, (p - 1)i$ we obtain the following matrix equation, remembering that

$$\mu_{0m} = (\mathbf{U}_0, \bar{\chi}_m) = N^{-1/2} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms},$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m1}^{i} & \xi_{m2}^{i} & \xi_{m3}^{i} & \cdots & \xi_{mp}^{i} \\ \xi_{m1}^{2i} & \xi_{m2}^{2i} & \xi_{m3}^{2i} & \cdots & \xi_{mp}^{2i} \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{m1}^{(p-1)i} & \xi_{m2}^{(p-1)i} & \xi_{m3}^{(p-1)i} & \cdots & \xi_{mp}^{(p-1)i} \end{bmatrix} \begin{bmatrix} \lambda_{m1} \\ \lambda_{m2} \\ \lambda_{m3} \\ \vdots \\ \lambda_{mp} \end{bmatrix} = \frac{1}{\sum_{s=0}^{N-1} u_{0s}\omega^{-ms}} \begin{bmatrix} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{is}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{2i,s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{(p-1)i,s}\omega^{-ms} \end{bmatrix}, \tag{4.20}$$

$$(m = 0, 1, 2, \cdots, (N - 1)).$$

If at the i-th step the radius of the circle is contracted to δ then by putting

$$\xi_{jk} = \rho^{\tau_{jk}}, \tag{4.21}$$

we have $\xi^i_{jk} = \rho^{i\tau_{jk}} = \delta^{\tau_{jk}}$. If we keep δ fixed and let $\rho \to 1$ and $N \to \infty$, Equation (4.20) becomes

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \delta^{\tau_{m1}} & \delta^{\tau_{m2}} & \delta^{\tau_{m3}} & \cdots & \delta^{\tau_{mp}} \\ \delta^{2\tau_{m1}} & \delta^{2\tau_{m2}} & \delta^{2\tau_{m3}} & \cdots & \delta^{2\tau_{mp}} \\ \vdots & \vdots & \vdots & & \vdots \\ \delta^{(p-1)\tau_{m1}} & \delta^{(p-1)\tau_{m2}} & \delta^{(p-1)\tau_{m3}} & \cdots & \delta^{(p-1)\tau_{mp}} \end{bmatrix} \begin{bmatrix} \lambda_{m1} \\ \lambda_{m2} \\ \lambda_{m3} \\ \vdots \\ \lambda_{mp} \end{bmatrix}$$

$$= \frac{1}{\int_0^{2\pi} u(1, \varphi)e^{-\iota m\varphi}\, d\varphi} \begin{bmatrix} \int_0^{2\pi} u(1, \varphi)e^{-\iota m\varphi}\, d\varphi \\ \int_0^{2\pi} u(\delta, \varphi)e^{-\iota m\varphi}\, d\varphi \\ \int_0^{2\pi} u(\delta^2, \varphi)e^{-\iota m\varphi}\, d\varphi \\ \vdots \\ \int_0^{2\pi} u(\delta^{p-1}, \varphi)e^{-\iota m\varphi}\, d\varphi \end{bmatrix}, \qquad (m = 0, 1, 2, \cdots). \tag{4.22}$$

Now τ_{m1} , τ_{m2} , $\cdots$, τ_{mp} are all different since by our assumption ξ_{m1} , ξ_{m2} , $\cdots$, τ_{mp} are all different, so that the matrix on the left-hand side of (4.22) is non-singular and λ_{m1} , λ_{m2} , $\cdots$, λ_{mp} are bounded.

The boundedness of λ_{jk} in the limit ensures c_{jk} to be bounded, from (4.17). We have already seen that $\xi^i_{jk} = \rho^{i\tau_{jk}} = \delta^{\tau_{jk}}$ which is bounded. Therefore, as given by (4.9) R_n is of the order of $\sum_{j=n+1}^{\infty} j^{-\nu}$ as $N \to \infty$ and $\rho \to 1$, and can be made arbitrarily small by increasing n provided $\nu \geqq 2$.

Now refer to (4.8) and consider the first n harmonics of u_{is} , i.e.,

$$\begin{aligned} u^*_{is} &= u_{is} - R_n \\ &= c_{01}\xi^i_{01} + c_{02}\xi^i_{02} + \cdots + c_{0p}\xi^i_{0p} \\ &\qquad + \sum_{j=1}^{n} (c_{j1}\xi^i_{j1} + c_{j2}\xi^i_{j2} + \cdots + c_{jp}\xi^i_{jp})e^{i\theta_s \iota} \\ &\qquad + \sum_{j=1}^{n} (c_{N-j,1}\xi^i_{N-j,1} + c_{N-j,2}\xi^i_{N-j,2} + \cdots + c_{N-j,p}\xi^i_{N-j,p})e^{-i\theta_s \iota}. \end{aligned} \tag{4.23}$$

We recall ξ_{j1} , ξ_{j2} , $\cdots$, ξ_{jp} are the p roots of the equation (3.11). If we substitute $A_{\alpha\beta}$ as given by Equation (2.6) into (3.11) we get

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} \frac{(-)^\alpha a_{\alpha\beta}(\alpha - \beta)!}{\rho^{\sigma\alpha-\beta}(1 - \rho)(1 - \rho^2) \cdots (1 - \rho^{\alpha-\beta})} \left[\left(\frac{1 - \omega^j}{\omega^{1/2j}\, \Delta\theta} \right)^\beta, \left(\frac{1 - \omega^j}{\omega^{1/2j}\, \Delta\theta} \right)^\beta \omega^{1/2j} \right]$$
$$\cdot(\xi_j - 1)(\xi_j - \rho) \cdots (\xi_j - \rho^{\alpha-\beta-1}) = 0, \qquad (j = 0, 1, 2, \cdots, n), \tag{4.24}$$

after cancelling out the factor $\omega^{jp/2}$, where the first or the second member in the bracket is to be taken according as β is even or odd. Since $\omega = \exp(2\pi\iota/N)$ and $\Delta\theta = 2\pi/N$ it follows by decreasing $\Delta\theta$ indefinitely,

$$\lim_{N\to\infty}\left(\frac{1-\omega^{j}}{\omega^{j/2}\,\Delta\theta}\right)^{\beta} = \lim_{N\to\infty}(-\iota)^{\beta}\left(\frac{N}{\pi}\sin\frac{\pi j}{N}\right)^{\beta} = (-j\iota)^{\beta}, \qquad (j = 0, 1, 2, \cdots, n). \tag{4.25}$$

Also, since $\lim_{N\to\infty}\omega^{j/2} = 1$, Equation (4.24) is reduced to the following equation as $N \to \infty$:

$$\sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}\frac{(-)^{\alpha}(-j\iota)^{\beta}a_{\alpha\beta}(\alpha-\beta)!}{\rho^{\sigma\alpha-\beta}(1-\rho)(1-\rho^{2})\cdots(1-\rho^{\alpha-\beta})}(\xi_j-1)(\xi_j-\rho)\cdots(\xi_j-\rho^{\alpha-\beta-1}) = 0,$$

$$(j = 0, 1, 2, \cdots, n). \tag{4.26}$$

We now shrink the mesh size radially by letting $\rho \to 1$. As before, putting

$$\xi_j = \rho^{\tau_j}, \qquad (j = 0, 1, 2, \cdots, n), \tag{4.27}$$

we have:

$$\lim_{\rho\to1}\frac{1}{\rho^{\sigma\alpha-\beta}}\frac{(\xi_j-1)(\xi_j-\rho)\cdots(\xi_j-\rho^{\alpha-\beta-1})}{(1-\rho)(1-\rho^{2})\cdots(1-\rho^{\alpha-\beta})}$$

$$= \lim_{\rho\to1}\frac{\xi_j-1}{1-\rho}\lim_{\rho\to1}\frac{\xi_j-\rho}{1-\rho^{2}}\cdots\lim_{\rho\to1}\frac{\xi_j-\rho^{\alpha-\beta-1}}{1-\rho^{\alpha-\beta}}$$

$$= \frac{\tau_j}{-1}\,\frac{\tau_j-1}{-2}\cdots\frac{\tau_j-(\alpha-\beta-1)}{-(\alpha-\beta)}$$

$$= \frac{\tau_j(\tau_j-1)\cdots(\tau_j-\alpha+\beta+1)}{(-)^{\alpha-\beta}(\alpha-\beta)!}. \tag{4.28}$$

Combining Equations (4.26) and (4.28) we find in the limit as $N \to \infty$ and then in the limit as $\rho \to 1$, τ_j satisfies the following equation:

$$\sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}a_{\alpha\beta}\tau_j(\tau_j-1)\cdots(\tau_j-\alpha+\beta+1)(j\iota)^{\beta} = 0, \quad (j = 0, 1, 2, \cdots, n). \tag{4.29}$$

We write τ_{-j} in place of τ_{N-j} as $N \to \infty$ (similarly $\xi_{-j,k}$, $C_{-j,k}$ for $\xi_{N-j,k}$, $C_{N-j,k}$) and find in like manner as $N \to \infty$ and $\rho \to 1$ τ_{-j} satisfies the equation:

$$\sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}a_{\alpha\beta}\tau_{-j}(\tau_{-j}-1)\cdots(\tau_{-j}-\alpha+\beta+1)(-j\iota)^{\beta} = 0, \quad (j = 0, 1, 2, \cdots, n). \tag{4.30}$$

Clearly from (4.29) and (4.30)

$$\tau_{jk} = \bar{\tau}_{-j,k}, \qquad (j = 1, 2, \cdots; k = 1, 2, \cdots, p), \tag{4.31}$$

and since $\xi_{jk} = \rho^{\tau_{jk}}$,

$$\xi_{jk} = \bar{\xi}_{-j,k}, \qquad (j = 1, 2, \cdots; k = 1, 2, \cdots, p). \tag{4.32}$$

We put $\xi_{jk} = \rho^{\tau_{jk}}$, $\rho^{i} = \delta$, δ being kept fixed and recall $u_{i,s}$ is the $(s+1)$th component of $\mathbf{U}_i$ i.e., $u(i, -p/2+s)$ or $u(i, -(p+1)/2+s)$ depending on p is even or odd according

(2.8). In the limit as $N \to \infty$ both $u(i, -p/2 + s)$ and $u(i, -(p+1)/2 + s)$ approach $u(\delta, \theta_s)$ then Equation (4.23) becomes

$$\begin{aligned} u^*(\delta_1 \theta_s) &= c_{01}\, \delta^{\tau_{01}} + c_{02}\, \delta^{\tau_{02}} + \cdots + c_{0p}\, \delta^{\tau_{0p}} \\ &\quad + \sum_{j=1}^{n} (c_{j1}\, \delta^{\tau_{j1}} + c_{j2}\, \delta^{\tau_{j2}} + \cdots + c_{jp}\, \delta^{\tau_{jp}}) \exp(j\theta_s \iota) \\ &\quad + \sum_{j=1}^{n} (c_{-j,1}\, \delta^{\tau_{-j,1}} + c_{-j,2}\, \delta^{\tau_{-j,2}} + \cdots + c_{-j,p}\, \delta^{\tau_{-j,p}}) \exp(-j\theta_s \iota). \end{aligned} \tag{4.33}$$

Now in the given differential equation (2.1), if we seek a solution in the form $r^{\sigma_i} \exp(j\theta\iota)$ or $r^{\sigma'_i} \exp(-j\theta\iota)$ we must have, by substituting them into the differential equation,

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} a_{\alpha\beta}\sigma_j(\sigma_j - 1) \cdots (\sigma_j - \alpha + \beta + 1)(j\iota)^{\beta} = 0, \qquad (j = 0, 1, 2, \cdots), \tag{4.34}$$

$$\sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} a_{\alpha\beta}\sigma'_j(\sigma'_j - 1) \cdots (\sigma'_j - \alpha + \beta + 1)(-j\iota)^{\beta} = 0, \qquad (j = 0, 1, 2, \cdots). \tag{4.35}$$

Comparing (4.29) with (4.34) and (4.30) with (4.35) we find

$$\tau_j = \sigma_j, \qquad \tau_{-j} = \sigma'_j, \qquad (j = 0, 1, 2, \cdots), \tag{4.36}$$

and Equation (4.36) indeed expresses the fact that the first n harmonics obtained by the boundary contraction method converge in the limit to the first n harmonics of the Fourier series expansion.

Part II Boundary Conditions

5. Computation stability. It has already been demonstrated that the general solution of the difference equation (2.11) as given by (3.16) involves Np arbitrary constants. These constants must be determined by the boundary conditions associated with the differential equation in such a way that a unique and bounded solution is defined in the region under consideration. In the following paragraphs it will be shown that this condition imposes a restraint upon the type of boundary conditions that may be specified.

From (3.13), we see $X_1, X_2, \cdots, X_p$ are diagonalized under the unitary transformation Q. Let the eigenvalues of the circulants $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p$ be denoted by $\lambda_{jk}(j = 0, 1, 2, \cdots, (N -); k = 1, 2, \cdots, p)$ then the circulant property of these matrices assures us:

$$\mathcal{E}_k = Q \operatorname{diag}(\lambda_{0k}, \lambda_{1k}, \cdots, \lambda_{N-1,k})\bar{Q}. \tag{5.1}$$

Substitution of (3.14) and (5.1) into (3.16) gives:

$$\begin{aligned} \mathbf{U}_i &= \sum_{k=1}^{p} \{\mathcal{E}_k Q \operatorname{diag}(\xi_{0k}^i, \xi_{1k}^i, \cdots, \xi_{N-1,k}^i)\bar{Q}\} \mathbf{U}_0 \\ &= Q \sum_{k=1}^{p} \operatorname{diag}(\lambda_{0k}\xi_{0k}^i, \lambda_{1k}\xi_{1k}^i, \cdots, \lambda_{N-1,k}\xi_{N-1,k}^i)\bar{Q}\mathbf{U}_0 . \end{aligned} \tag{5.2}$$

It follows, therefore, that in order that the computation be stable, the solution as given by (5.2) should be bounded interior to the unit circle, i.e.,

$$| \xi_{jk} | \leqq 1, \qquad \begin{bmatrix} j = 0, 1, 2, \cdots, (N-1) \\ k = 1, 2, 3, \cdots, p \end{bmatrix}. \tag{5.3}$$

Some differential equations, when approximated by certain difference schemes, do yield eigenvalues satisfying (5.3), [5]. However, this is in general not true, and to ensure computational stability it is necessary to adjust the arbitrary matrices $\mathcal{E}_1, \mathcal{E}_2, \cdots \mathcal{E}_p$ so that for those ξ^i_{jk}'s which are larger than unity in absolute value, the corresponding λ_{jk}'s are zero. This reduces the arbitrariness of the matrices $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p$ and since they are determined by the given boundary conditions, this in turn restricts the arbitrariness of the boundary conditions. We proceed to deduce the conditions that the boundary data have to satisfy so that the solution will be bounded within the unit circle and computation will be stable as $r \to 0$.

Referring back to (4.20) we put $i = 1$ and obtain

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m1} & \xi_{m2} & \xi_{m3} & \cdots & \xi_{mp} \\ \xi^2_{m1} & \xi^2_{m2} & \xi^2_{m3} & \cdots & \xi^2_{mp} \\ \vdots & \vdots & \vdots & & \vdots \\ \xi^{p-1}_{m1} & \xi^{p-1}_{m2} & \xi^{p-1}_{m3} & \cdots & \xi^{p-1}_{mp} \end{bmatrix} \begin{bmatrix} \lambda_{m1} \\ \lambda_{m2} \\ \lambda_{m3} \\ \vdots \\ \lambda_{mp} \end{bmatrix}$$

$$= \frac{1}{\sum_{s=0}^{N-1} u_{0s}\omega^{-ms}} \begin{bmatrix} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{1s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{2s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{p-1,s}\omega^{-ms} \end{bmatrix}, \qquad (m = 0, 1, 2, \cdots, (N-1)). \tag{5.4}$$

Equation (5.4) expresses analytically the fact that eigenvalues of $\mathcal{E}_1, \mathcal{E}_2, \cdots \mathcal{E}_p$ i.e., $\lambda_{mk}(m = 0, 1, 2, \cdots, (N-1); k = 1, 2, 3, \cdots, p)$ are determined by the boundary data on p consecutive circles: $\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_{p-1}$. To ensure the solution to remain bounded in the region considered so that computation will be stable, we must discard those ξ_{mk}'s for which the absolute values are larger than unity. Suppose that we arrange $\xi_{mk}(m = 0, 1, 2, \cdots, (N-1); k = 1, 2, 3, \cdots, p)$ according to magnitude such that

$$| \xi_{m1} | \leqq 1, \qquad | \xi_{m2} | \leqq 1, \cdots, | \xi_{mq_m} | \leqq 1$$

and

$$| \xi_{m,q_m+1} | > 1, \quad | \xi_{m,q_m+2} | > 1, \cdots, | \xi_{mp} | > 1, \quad (m = 0, 1, 2, \cdots, (N-1)), \tag{5.5}$$

then, we must discard $\xi_{m,q_m+1}, \xi_{m,q_m+2}, \cdots, \xi_{mp}$ by putting

$$\lambda_{m,q_m+1} = \lambda_{m,q_m+2} = \cdots = \lambda_{mp} = 0. \tag{5.6}$$

This results

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m1} & \xi_{m2} & \xi_{m3} & \cdots & \xi_{mq_m} \\ \xi_{m1}^2 & \xi_{m2}^2 & \xi_{m3}^2 & \cdots & \xi_{mq_m}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{m1}^{p-1} & \xi_{m2}^{p-1} & \xi_{m3}^{p-1} & \cdots & \xi_{mq_m}^{p-1} \end{bmatrix} \begin{bmatrix} \lambda_{m1} \\ \lambda_{m2} \\ \lambda_{m3} \\ \vdots \\ \lambda_{mq_m} \end{bmatrix}$$

$$= \frac{1}{\sum_{s=0}^{N-1} u_{0s}\omega^{-ms}} \begin{bmatrix} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{1s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{2s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{p-1,s}\omega^{-ms} \end{bmatrix}, \qquad (m = 0, 1, 2, \cdots, (N-1)). \tag{5.7}$$

In the matrix equation (5.7) $\lambda_{m1}, \lambda_{m2}, \cdots, \lambda_{mq_m}$ can be eliminated by solving the first q_m equations for $\lambda_{m1}, \lambda_{m2}, \cdots, \lambda_{mq_m}$ and substituting the result into the remaining $p - q_m$ equations. This is possible if $\xi_{m1}, \xi_{m2}, \cdots, \xi_{mq_m}$ are all different. Assuming this to be true, we obtain the eliminant as follows:

$$\begin{bmatrix} \xi_{m1}^{q_m} & \xi_{m2}^{q_m} & \xi_{m3}^{q_m} & \cdots & \xi_{mq_m}^{q_m} \\ \xi_{m1}^{q_m+1} & \xi_{m2}^{q_m+1} & \xi_{m3}^{q_m+1} & \cdots & \xi_{mq_m}^{q_m+1} \\ \xi_{m1}^{q_m+2} & \xi_{m2}^{q_m+2} & \xi_{m3}^{q_m+2} & \cdots & \xi_{mq_m}^{q_m+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{m1}^{p-1} & \xi_{m2}^{p-1} & \xi_{m3}^{p-1} & \cdots & \xi_{mq_m}^{p-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m1} & \xi_{m2} & \xi_{m3} & \cdots & \xi_{mq_m} \\ \xi_{m1}^2 & \xi_{m2}^2 & \xi_{m3}^2 & \cdots & \xi_{mq_m}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{m1}^{q_m-1} & \xi_{m2}^{q_m-1} & \xi_{m3}^{q_m-1} & \cdots & \xi_{mq_m}^{q_m-1} \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{1s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{2s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{q_m-1,s}\omega^{-ms} \end{bmatrix} = \begin{bmatrix} \sum_{s=0}^{N-1} u_{q_m s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{q_m+1,s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{q_m+2,s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{p-1,s}\omega^{-ms} \end{bmatrix}, \qquad (m = 0, 1, 2, \cdots, (N-1)) \tag{5.8}$$

or

$$\Psi_m \Phi_m^{-1} \mathbf{W}_{m1} = \mathbf{W}_{m2}, \qquad (m = 0, 1, 2, \cdots, (N-1)), \tag{5.9}$$

where the symbols have the obvious meaning.

6. Evaluation of Φ_m^{-1}. To find Φ_m^{-1} we consider the following expression in η:

$$\sum_{j=1}^{q_m} \frac{(\eta - \xi_{m1})(\eta - \xi_{m2}) \cdots (\eta - \xi_{m,j-1})(\eta - \xi_{m,j+1}) \cdots (\eta - \xi_{mq_m})}{(\xi_{mj} - \xi_{m1})(\xi_{mj} - \xi_{m2}) \cdots (\xi_{mj} - \xi_{m,j-1})(\xi_{mj} - \xi_{m,j+1}) \cdots (\xi_{mj} - \xi_{mq_m})} \xi_{mj}^i = \eta^i,$$

$$(i = 0, 1, 2, \cdots, q_m - 1;\ m = 0, 1, 2, \cdots, N - 1). \tag{6.1}$$

Equation (6.1) is clearly satisfied by q_m different values of $\eta : \xi_{m1}, \xi_{m2}, \cdots, \xi_{mq_m}$ but it is at most of the degree $q_m - 1$ in η; hence it must be an identity. Let

$$\begin{aligned}(\eta - \xi_{m1})(\eta - \xi_{m2}) \cdots (\eta - \xi_{m,j-1})(\eta - \xi_{m,j+1}) \cdots (\eta - \xi_{mq_m}) \\ = \eta^{q_m-1} + t_{mj1}\eta^{q_m-2} + t_{mj2}\eta^{q_m-3} + \cdots + t_{mj,q_m-1},\end{aligned} \tag{6.2}$$

$$\begin{aligned}(\xi_{mj} - \xi_{m1})(\xi_{mj} - \xi_{m2}) \cdots (\xi_{mj} - \xi_{m,j-1})(\xi_{mj} - \xi_{m,j+1}) \cdots (\xi_{mj} - \xi_{mq_m}) \\ = \prod_{\epsilon=1}^{q_m}{}' (\xi_{mj} - \xi_{m\epsilon}).\end{aligned} \tag{6.3}$$

Then by substituting (6.2) and (6.3) into (6.1), and comparing the coefficients of $\eta^0, \eta^1, \eta^2, \cdots, \eta^{q_m-1}$, we obtain

$$\begin{cases} \displaystyle\sum_{j=1}^{q_m} \frac{t_{mj,q_m-1}\xi_{mj}^i}{\prod_{\epsilon=1}^{q_m}{}' (\xi_{mj} - \xi_{m\epsilon})} = \delta_{i0}, \\ \displaystyle\sum_{j=1}^{q_m} \frac{t_{mj,q_m-2}\xi_{mj}^i}{\prod_{\epsilon=1}^{q_m}{}' (\xi_{mj} - \xi_{m\epsilon})} = \delta_{i1}, \\ \cdots \\ \displaystyle\sum_{j=1}^{q_m} \frac{\xi_{mj}^i}{\prod_{\epsilon=1}^{q_m}{}' (\xi_{mj} - \xi_{m\epsilon})} = \delta_{i,q_m-1}, \qquad \begin{bmatrix} i = 0, 1, 2, \cdots, q_m - 1, \\ m = 0, 1, 2, \cdots, N - 1 \end{bmatrix} \end{cases} \tag{6.4}$$

where $\delta_{i0}, \delta_{i1}, \cdots$ are the Kronecker deltas. From the relation (6.4) we conclude

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m1} & \xi_{m2} & \xi_{m3} & \cdots & \xi_{mq_m} \\ \xi_{m1}^2 & \xi_{m2}^2 & \xi_{m3}^2 & \cdots & \xi_{mq_m}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{m1}^{q_m-1} & \xi_{m2}^{q_m-1} & \xi_{m3}^{q_m-1} & \cdots & \xi_{mq_m}^{q_m-1} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{t_{m1,q_m-1}}{\prod_\epsilon' (\xi_{m1} - \xi_{m\epsilon})} & \dfrac{t_{m1,q_m-2}}{\prod_\epsilon' (\xi_{m1} - \xi_{m\epsilon})} & \dfrac{t_{m1,q_m-3}}{\prod_\epsilon' (\xi_{m1} - \xi_{m\epsilon})} & \cdots & \dfrac{1}{\prod_\epsilon' (\xi_{m1} - \xi_{m\epsilon})} \\ \dfrac{t_{m2,q_m-1}}{\prod_\epsilon' (\xi_{m2} - \xi_{m\epsilon})} & \dfrac{t_{m2,q_m-2}}{\prod_\epsilon' (\xi_{m2} - \xi_{m\epsilon})} & \dfrac{t_{m2,q_m-3}}{\prod_\epsilon' (\xi_{m2} - \xi_{m\epsilon})} & \cdots & \dfrac{1}{\prod_\epsilon' (\xi_{m2} - \xi_{m\epsilon})} \\ \dfrac{t_{m3,q_m-1}}{\prod_\epsilon' (\xi_{m3} - \xi_{m\epsilon})} & \dfrac{t_{m3,q_m-2}}{\prod_\epsilon' (\xi_{m3} - \xi_{m\epsilon})} & \dfrac{t_{m3,q_m-3}}{\prod_\epsilon' (\xi_{m3} - \xi_{m\epsilon})} & \cdots & \dfrac{1}{\prod_\epsilon' (\xi_{m3} - \xi_{m\epsilon})} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{t_{mq_m,q_m-1}}{\prod_\epsilon' (\xi_{mq_m} - \xi_{m\epsilon})} & \dfrac{t_{mq_m,q_m-2}}{\prod_\epsilon' (\xi_{mq_m} - \xi_{m\epsilon})} & \dfrac{t_{mq_m,q_m-3}}{\prod_\epsilon' (\xi_{mq_m} - \xi_{m\epsilon})} & \cdots & \dfrac{1}{\prod_\epsilon' (\xi_{mq_m} - \xi_{m\epsilon})} \end{bmatrix},$$

$$(m = 0, 1, 2, \cdots, (q_m - 1)). \tag{6.5}$$

7. Evaluation of $\Psi_m \Phi_m^{-1}$. It remains to evaluate the product $\Psi_m \Phi_m^{-1}$. To this end we consider the following expression:

$$F(\eta) = \sum_{j=1}^{q_m} \frac{(\eta - \xi_{m1})(\eta - \xi_{m2}) \cdots (\eta - \xi_{m,j-1})(\eta - \xi_{m,j+1}) \cdots (\eta - \xi_{mq_m})}{(\xi_{mj} - \xi_{m1})(\xi_{mj} - \xi_{m2}) \cdots (\xi_{mj} - \xi_{m,j-1})(\xi_{mj} - \xi_{m,j+1}) \cdots (\xi_{mj} - \xi_{mq_m})} \xi_{mj}^i - \eta^i,$$

$$\begin{bmatrix} i = q_m, q_m + 1, \cdots, p - 1 \\ m = 0, 1, 2, \cdots, N - 1 \end{bmatrix}. \tag{7.1}$$

Clearly $F(\xi_{m1}) = F(\xi_{m2}) = \cdots = F(\xi_{mq_m}) = 0$ and since $F(\eta)$ is of degree equal to or higher than q_m in η, it is divisible by the factor

$$(\eta - \xi_{m1})(\eta - \xi_{m2}) \cdots (\eta - \xi_{mq_m}) = \eta^{q_m} + t_{m1}\eta^{q_m-1} + t_{m2}\eta^{q_m-2} + \cdots + t_{mq_m}, \quad \text{say.} \tag{7.2}$$

We can therefore write

$$\sum_{j=1}^{q_m} \frac{(\eta - \xi_{m1})(\eta - \xi_{m2}) \cdots (\eta - \xi_{m,j-1})(\eta - \xi_{m,j+1}) \cdots (\eta - \xi_{mq_m})}{(\xi_{mj} - \xi_{m1})(\xi_{mj} - \xi_{m2}) \cdots (\xi_{mj} - \xi_{m,j-1})(\xi_{mj} - \xi_{m,j+1}) \cdots (\xi_{mj} - \xi_{mq_m})} \xi_{mj}^i = \eta^i$$
$$+ (\eta^{q_m} + t_{m1}\eta^{q_m-1} + t_{m2}\eta^{q_m-2} + \cdots + t_{mq_m})(c_{im0} + c_{im1}\eta$$
$$+ c_{im2}\eta^2 + \cdots + c_{im,i-q_m}\eta^{i-q_m}), \qquad \begin{bmatrix} i = q_m, q_m + 1, \cdots, p - 1 \\ m = 0, 1, 2, \cdots, N - 1 \end{bmatrix}, \tag{7.3}$$

and by using (7.2) and (7.3),

$$\sum_{j=1}^{q_m} \frac{\eta^{q_m-1} + t_{mj1}\eta^{q_m-2} + t_{mj2}\eta^{q_m-3} + \cdots + t_{mj,q_m-1}}{\prod_{\epsilon=1}^{q_m}{}' (\xi_{mj} - \xi_{m\epsilon})} \xi_{mj}^i = \eta^i(c_{im,i-q_m} + 1)$$
$$+ \eta^{i-1}(t_{m1}c_{im,i-q_m} + c_{im,i-q_m-1})$$
$$+ \eta^{i-2}(t_{m2}c_{im,i-q_m} + t_{m1}c_{im,i-q_m-1} + c_{im,i-q_m-2})$$
$$+ \eta^{i-3}(t_{m3}c_{im,i-q_m} + t_{m2}c_{im,i-q_m-1} + t_{m1}c_{im,i-q_m-2} + c_{im,i-q_m-3})$$
$$+ \cdots$$
$$+ \eta^2(t_{mq_m}c_{im2} + t_{m,q_m-1}c_{im1} + t_{m,q_m-2}c_{im0})$$
$$+ \eta(t_{mq_m}c_{im1} + t_{m,q_m-1}c_{im0})$$
$$+ t_{mq_m}c_{im0}. \tag{7.4}$$

Comparing the coefficients of $\eta^0, \eta^1, \eta^2, \cdots, \eta^{q_m-1}$ on both sides of (7.4) we obtain the following expressions:

$$\sum_{j=1}^{q_m} \frac{t_{mj,q_m-1}\xi_{mj}^i}{\prod_{\epsilon}' (\xi_{mj} - \xi_{m\epsilon})} = t_{mq_m}c_{im0},$$

$$\sum_{j=1}^{q_m} \frac{t_{mj,q_m-2}\xi_{mj}^i}{\prod_{\epsilon}' (\xi_{mj} - \xi_{m\epsilon})} = t_{m,q_m-1}c_{im0} + t_{mq_m}c_{im1}I(q_m + 1),$$

$$\sum_{j=1}^{q_m} \frac{t_{mj,q_m-3}\xi_{mj}^i}{\prod_\epsilon' (\xi_{mj} - \xi_{m\epsilon})} = t_{m,q_m-2}c_{im0} + t_{m,q_m-1}c_{im1}I(q_m + 1) + t_{mq_m}c_{im2}I(q_m + 2),$$

$$\cdots$$

$$\sum_{j=1}^{q_m} \frac{\xi_{mj}^i}{\prod_\epsilon' (\xi_{mj} - \xi_{m\epsilon})} = t_{m1}c_{im0} + t_{m2}c_{im1}I(q_m + 1) + \cdots + t_{mq_m}c_{im,q_m-1}I(2q_m - 1),$$

$$\begin{bmatrix} i = q_m\,, q_m + 1, \cdots, p - 1 \\ m = 0, 1, 2, \cdots, N - 1 \end{bmatrix}, \tag{7.5}$$

where

$$I(x) = 1, \qquad i \geqq x, \\ = 0, \qquad i < x. \tag{7.6}$$

Similarly, by equating coefficients of η^i, η^{i-1}, η^{i-2}, $\cdots$, η^{q_m} we obtain the following set of equations:

$$\begin{aligned}
&c_{im,i-q_m} + 1 = 0, && p - 1 \geqq i \geqq q_m\,, \\
&t_{m1}c_{im,i-q_m} + c_{im,i-q_m-1} = 0, && p - 1 \geqq i \geqq q_m + 1, \\
&t_{m2}c_{im,i-q_m} + t_{m1}c_{im,i-q_m-1} + c_{im,i-q_m-2} = 0, && p - 1 \geqq i \geqq q_m + 2, \\
&\cdots \\
&t_{mq_m}c_{im,i-q_m} + t_{m,q_m-1}c_{im,i-q_m-1} + \cdots \\
&\qquad + t_{m1}c_{im,i-2q_m+1} + c_{im,i-2q_m} = 0, && p - 1 \geqq i \geqq 2q_m\,, \\
&\cdots .
\end{aligned} \tag{7.7}$$

Thus, for $i = q_m$, only the first equation is valid; for $i = q_m + 1$, the first two equations are valid, etc.

From Equations (6.5) with (7.5) we obtain

$$\Psi_m\Phi_m^{-1} = \left[\begin{array}{ll}
t_{mq_m}c_{q_mm0}\,, & t_{m,q_m-1}c_{q_mm0}\,, \\
t_{mq_m}c_{q_m+1,m0}\,, & t_{m,q_m-1}c_{q_m+1,m0} + t_{mq_m}c_{q_m+1,m1}\,, \\
t_{mq_m}c_{q_m+2,m0}\,, & t_{m,q_m-1}c_{q_m+2,m0} + t_{mq_m}c_{q_m+2,m1}\,, \\
 & \cdots\cdots\cdots\cdots \\
 & \cdots\cdots\cdots\cdots \\
t_{mq_m}c_{p-1,m0}\,, & t_{mq_m}c_{p-1,m0} + t_{mq_m}c_{p-1,m1}\,,
\end{array}\right.$$

$$\begin{array}{l}
t_{m,q_m-2}c_{q_mm0}\,, \cdots\cdots\cdots\cdots \\
t_{m,q_m-2}c_{q_m+1,m0} + t_{m,q_m-1}c_{q_m+1,m1}\,, \cdots\cdots\cdots \\
t_{m,q_m-2}c_{q_m+2,m0} + t_{m,q_m-1}c_{q_m+2,m1} + t_{mq_m}c_{q_m+2,m2} \cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
t_{m,q_m-2}c_{p-1,m0} + t_{m,q_m-1}c_{p-1,m1} + t_{mq_m}c_{p-1,m2}\,, \cdots\cdots\cdots
\end{array}$$

$$\left.\begin{array}{ll}
\cdots, t_{m2}c_{q_mm0}\,, & t_{m1}c_{q_mm0} \\
\cdots, t_{m2}c_{q_m+1,m0} + t_{m3}c_{q_m+1,m1}\,, & t_{m1}c_{q_m+1,m0} + t_{m2}c_{q_m+1,m1} \\
\cdots, t_{m3}c_{q_m+2,m0} + t_{m3}c_{q_m+2,m1} + t_{m4}c_{q_m+2,m2}\,, & t_{m1}c_{q_m+2,m0} + t_{m2}c_{q_m+2,m1} + t_{m3}c_{q_m+2,m2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right]$$

$$
= \begin{bmatrix}
c_{q_m m0} & 0 & \cdots & \cdots & 0 \\
c_{q_m+1,m0} & c_{q_m+1,m1} & 0 & \cdots & 0 \\
c_{q_m+2,m0} & c_{q_m+2,m1} & c_{q_m+2,m2} & 0 \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & 0 \\
c_{p-1,m0} & c_{p-1,m1} & c_{p-1,m2} & \cdots & c_{p-1,m,p-q_m-1}
\end{bmatrix}
\cdot \begin{bmatrix}
t_{mq_m} & t_{m,q_m-1} & t_{m,q_m-2} & \cdots & t_{m3} & t_{m2} & t_{m1} \\
0 & t_{mq_m} & t_{m,q_m-1} & \cdots & t_{m4} & t_{m3} & t_{m2} \\
\vdots & 0 & t_{mq_m} & \cdots & t_{m5} & t_{m4} & t_{m3} \\
\vdots & & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 \cdots & & t_{m,p-q_m-1} & t_{m,p-q_m}
\end{bmatrix} \tag{7.8}
$$

On the other hand, by assigning $i = q_m, q_m + 1, q_m + 2, \cdots, p - 1$ in Equation (7.7) we can write the resulting equations in the following matrix form:

$$
\begin{bmatrix}
c_{q_m m0} & 0 & \cdots & \cdots & 0 \\
c_{q_m+1,m0} & c_{q_m+1,m1} & 0 & \cdots & 0 \\
c_{q_m+2,m0} & c_{q_m+2,m1} & c_{q_m+2,m2} & 0 \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & 0 \\
c_{p-1,m0} & c_{p-1,m1} & \cdots & \cdots & c_{p-1,m,p-q_m-1}
\end{bmatrix}
\cdot \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
t_{m1} & 1 & 0 & & & \vdots \\
t_{m2} & t_{m1} & 1 & & & \vdots \\
t_{m3} & t_{m2} & t_{m1} & & & \vdots \\
\vdots & \vdots & \vdots & & & 0 \\
t_{m,p-q_m-1} & t_{m,p-q_m-2} & t_{m,p-q_m-3} & \cdots & t_{m1} & 1
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 \cdots & 0 \\
0 & -1 & \vdots \\
\vdots & & \vdots \\
\vdots & & 0 \\
0 \cdots & 0 & -1
\end{bmatrix}. \tag{7.9}
$$

It follows from Equations (7.8) and (7.9) by eliminating the matrix involving c's that

$$\Psi_m\Phi_m^{-1} = (-)\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ t_{m1} & 1 & 0 & & & \vdots \\ t_{m2} & t_{m1} & 1 & & & \vdots \\ t_{m3} & t_{m2} & t_{m1} & & & \vdots \\ \vdots & \vdots & \vdots & & & 0 \\ t_{m,p-q_m-1} & t_{m,p-q_m-2} & t_{m,p-qm-3} & \cdots & t_{m1} & 1 \end{bmatrix}^{-1}$$

$$\cdot\begin{bmatrix} t_{mq_m} & t_{m,q_m-1} & \cdots & t_{m3} & t_{m2} & t_{m1} \\ 0 & t_{mq_m} & \cdots & t_{m4} & t_{m3} & t_{m2} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 \cdots 0 & t_{mq_m} & \cdots & & & t_{m,p-q_m} \end{bmatrix} = (-)\Omega_{m2}^{-1}\Omega_{m1}, \tag{7.10}$$

with obvious meanings of Ω_{m1} and Ω_{m2} .

8. Conclusions. Substitution of Equation (7.10) into Equation (5.9) gives the following result:

$$-\Omega_{m2}^{-1}\Omega_{m1}\mathbf{W}_{m1} = \mathbf{W}_{m2},$$

$$\Omega_{m1}\mathbf{W}_{m1} + \Omega_{m2}\mathbf{W}_{m2} = 0,$$

$$[\Omega_{m1}, \Omega_{m2}]\begin{bmatrix}\mathbf{W}_{m1} \\ \mathbf{W}_{m2}\end{bmatrix} = 0,$$

i.e.,

$$\Omega_m\mathbf{W}_m = 0, \qquad (m = 0, 1, 2, \cdots, (N-1)), \tag{8.1}$$

where Ω_m is a $(p - q_m) \times p$ matrix formed by the first $p - q_m$ rows of the $p \times p$ circulant: $C(t_{mqm}, t_{m,qm-1}, \cdots, t_{m2}, t_{m1}, 1, 0 \cdots 0)$.

$$\Omega_m = \begin{bmatrix} t_{mq_m} & t_{m,q_m-1} & \cdots & t_{m2} & t_{m1} & 1 & 0 & \cdots & \cdots & 0 \\ 0 & t_{mq_m} & \cdots & t_{m3} & t_{m2} & t_{m1} & 1 & 0 & & \vdots \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & 0 \\ 0 \cdots 0 & t_{mq_m} & \cdots & & & \cdots & t_{m3} & t_{m2} & t_{m1} & 1 \end{bmatrix}, \tag{8.2}$$

and

$$\mathbf{W}_m = \begin{bmatrix} \sum_{s=0}^{N-1} u_{0s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{1s}\omega^{-ms} \\ \sum_{s=0}^{N-1} u_{2s}\omega^{-ms} \\ \vdots \\ \sum_{s=0}^{N-1} u_{p-1,s}\omega^{-ms} \end{bmatrix}. \tag{8.3}$$

Equation (8.1) is the bounded part of the solution of the difference equations which approximate Equation (2.1); it expresses analytically the fact that if the solution $u(r, \theta)$ is bounded in the unit circle, the values of $u(r, \theta)$ on any p consecutive circles $C_\alpha, C_{\alpha+1}, C_{\alpha+2}, \cdots, C_{\alpha+p-1}$ have to satisfy the relations expressed by (8.1). The total number of these relations is $(p - q_0) + (p - q_1) + \cdots + (p - q_{N-1}) = Np - (q_0 + q_1 + \cdots + q_{N-1})$. Therefore, if the solution is to remain bounded and computation to be stable as $r \to 0$, the given data on the circles $C_0, C_1, C_2, \cdots, C_{p-1}$ have to satisfy these relations. Since there are Np points $C_0, C_1, C_2, \cdots, C_{p-1}$, the boundary data cannot be specified at will: in fact, only $q_0 + q_1 + q_2 + \cdots + q_{N-1}$ points can be assigned arbitrary values.

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