AN EXTENSION OF 'SEPARATION-OF-VARIABLES' FOR TIME-DEPENDENT EXCITATIONS*

By I. U. OJALVO (*Republic Aviation Corp., Farmingdale, L. I., N. Y.*

1. Introduction. A general form of a procedure for extending the Separation-of-Variables technique to problems with time-dependent forcing-functions and boundary-conditions is presented. While the procedure originated and developed through the efforts of researchers in elastic vibrations [1, 2, 3, 4], the method is applicable to many linear partial differential equations which are operative in finite domains. Chow [5] has developed the method independently for second order differential equations involving only one independent space variable, and the approach has recently been extended to help solve the heat-conduction equation [6]. The method has received little attention outside the vibration field, although its generality has been recognized by others, as is evidenced by the following statement by Mindlin and Goodman [2]: "The method is here developed for and applied to the problem of the flexural vibrations of beams. It is equally applicable to time-dependent boundary-value problems ⋯ in a wide variety of systems governed by linear partial-differential equations." Furthermore, the present author is not aware of a treatment of this method in a form as general as is presented here.

2. The method. Consider the equation

$$Mu(P, t) = TNu(P, t) + F \quad \text{in} \quad D \tag{1}$$

with linear boundary condition

$$\alpha u + \beta u' + \gamma u'' + \cdots = G \quad \text{on} \quad B, \qquad t > 0 \tag{2}$$

and initial conditions

$$u = H_1(P), \qquad \partial u/\partial t = H_2(P) \quad \text{for } P \text{ in } D \text{ with } t = 0. \tag{3}$$

D is a continuum domain with boundaries on B. P and t denote the independent variables (space and time). M and N represent linear partial-differential space operators and T is the time operator

$$T = a\, \partial^2(\ )/\partial t^2 + b\, \partial(\ )/\partial t.$$

The boundary condition (2) may involve linear combinations of the space derivative of any order less than the highest ordered derivative appearing in M, and the order of M is greater than that of N.

Assume that a complete set of characteristic-value solutions φ_n associated with D and B are obtainable from the eigenvalue problem defined by

$$M\varphi_n = \lambda_n N\varphi_n \quad \text{in} \quad D, \tag{4.a}$$

$$\alpha\varphi_n + \beta\varphi_n' + \gamma\varphi_n'' + \cdots = 0 \quad \text{on} \quad B. \tag{4.b}$$

*Received December 7, 1961; revised manuscript received May 29, 1962.

Assume further that F and G of (1) and (2) are of the form

$$F = F(P, t) = \sum_{i=1}^{\mu} f_i(P)k_i(t), \tag{5}$$

$$G = G(P, t) = \sum_{j=1}^{\nu} g_j(P)q_j(t). \tag{6}$$

Assume a solution to (1) through (3) to be of the form

$$u = \sum_n \varphi_n(P)\psi_n(t) + \sum_i v_i(P)k_i(t) + \sum_j w_j(P)q_j(t). \tag{7}$$

Substitution of (5) and (7) into (1) yields

$$\sum_n \psi_n M\varphi_n + \sum_i k_i(Mv_i - f_i) + \sum_j q_j M w_j$$
$$= \sum_n T\psi_n N\varphi_n + \sum_i Tk_i N v_i + \sum_j Tq_j N w_j . \tag{8}$$

To simplify (8), let the v_i and w_j satisfy the corresponding statical equations

$$Mv_i(P) = f_i(P), \tag{9}$$

$$Mw_j(P) = 0. \tag{10}$$

Equation (8) then reduces to

$$\sum_n \psi_n M\varphi_n = \sum_n T\psi_n N\varphi_n + \sum_i Tk_i N v_i + \sum_j Tq_j N w_j . \tag{11}$$

To complete the separation of variables in (11), use the functions φ_n as a basis for the expansion of the v_i and w_j , i.e. let

$$v_i(P) = \sum_n c_{in}\varphi_n(P), \tag{12.a}$$

$$w_j(P) = \sum_n e_{jn}\varphi_n(P). \tag{12.b}$$

Substituting (12.a) and (12.b) into (11), equating termwise in n and dividing through by $\psi_n(t)N\varphi_n(P)$, we obtain the typical equation

$$M\varphi_n/N\varphi_n = (T\psi_n + \sum_i c_{in}Tk_i + \sum_j e_{jn}Tq_j)/\psi_n = -a\omega_n^2 \tag{13}$$

in which $-a\omega_n^2$ has been chosen as the separation constant. Comparing (4.a) with (13) we find

$$-\omega_n^2 = \lambda_n/a. \tag{14}$$

Equations for determining the ψ_n follow from (13):

$$\partial^2\psi_n/\partial t^2 + b/a\ \partial\psi_n/\partial t + \omega_n^2\psi_n = -(1/a)T(\sum_i c_{in}k_i + \sum_j e_{jn}q_j). \tag{16}$$

In a similar way the assumed solution form (7) is substituted into the boundary condition (2) and the initial conditions (3) to give

$$\sum_n \psi_n(\alpha\varphi_n + \beta\varphi_n' + \gamma\varphi_n'' + \cdots) + \sum_i k_i(\alpha v_i + \beta v_i' + \cdots)$$
$$+ \sum_j q_j((\alpha w_j + \beta w_j' + \cdots) - g_j) = 0 \quad \text{on} \quad B, \tag{17}$$

$$\sum_n \psi_n\varphi_n + \sum_i k_i v_i + \sum_j q_j w_j = H_1 \quad \text{for} \quad t = 0, \tag{18}$$

$$\sum_n (\partial\psi_n/\partial t)\varphi_n + \sum_i (\partial k_i/\partial t)v_i + \sum_j (\partial q_j/\partial t)w_j = H_2 \quad \text{for} \quad t = 0. \tag{19}$$

Since the k_i and q_j are independent functions of t, their coefficients must vanish in (17). Thus, together with (4.b) take

$$\alpha v_i + \beta v_i' + \gamma v_i'' + \cdots = 0 \quad \text{on} \quad B, \tag{20}$$

$$\alpha w_j + \beta w_j' + \cdots = g_j \quad \text{on} \quad B. \tag{21}$$

To satisfy the transformed initial conditions (18) and (19), first expand $H_1(P)$ and $H_2(P)$ into eigenvector series; i.e. let

$$H_1(P) = \sum_n h_{1n}\varphi_n(P), \tag{22}$$

$$H_2(P) = \sum_n h_{2n}\varphi_n(P). \tag{23}$$

Substitute (12.a), (12.b), (22) and (23) into (18) and (19) and rearrange to get

$$\psi_n(0) = h_{in} - (\sum_i c_{in}k_i(0) + \sum_j e_{jn}q_j(0)), \tag{24}$$

$$\partial\psi_n(0)/\partial t = h_{2n} - (\sum_i c_{in}\,\partial k_i(0)/\partial t + \sum_j e_{jn}\,\partial q_j(0)/\partial t). \tag{25}$$

To recapitulate, it has been shown how the original problem, as stated by (1), (2) and (3), may be transformed into a series of standard type subproblems defined by (4.a) and (4.b), (16), (24), and (25), (9) and (20), (10) and (21).

3. Orthogonality. To obtain the coefficients c_{in}, e_{jn}, h_{1n}, and h_{2n}, orthogonality properties are established. Consider (4.a) multiplied by a new space variable ξ_m and integrated over D, i.e.

$$\int_D \xi_m M\varphi_n\, dV_p = \lambda_n \int_D \xi_m N\varphi_n\, dV_p\,. \tag{26}$$

Define ξ_m as the solution to the adjoint equation corresponding to (4.a):

$$M^*\xi_m = \lambda_m^* N^*\xi_m \tag{27}$$

in which M^*, N^* and boundary conditions on ξ_m are such that

$$\int_D \xi_m N\varphi_n\, dV_p = \int_D \varphi_n N^*\xi_m\, dV_p\,, \tag{28}$$

$$\int_D \xi_m M\varphi_n\, dV_p = \int_D \varphi_n M^*\xi_m\, dV_p\,. \tag{29}$$

Multiplying (27) by φ_n and integrating over D, we find

$$\int_D \varphi_n M^* \xi_m \, dV_p = \lambda_m^* \int_D \varphi_n N^* \xi_m \, dV_p \,. \tag{30}$$

we now subtract (30) from (26) to get

$$\int_D \xi_m M \varphi_n \, dV_p - \int_D \varphi_n M^* \xi_m \, dV_p = \lambda_n \int_D \xi_m N \varphi_n \, dV_p - \lambda_m^* \int_D \varphi_n N^* \xi_m \, dV_p \,. \tag{31}$$

Applying (28) and (29) to (31), one obtains

$$(\lambda_n - \lambda_m^*) \int_D \varphi_n N^* \xi_m \, dV_p = 0. \tag{32}$$

It can be shown * that if λ_m^* is an eigenvalue of M^*, N^*, it is also an eigenvalue of M, N. Thus, (32) supplies the orthogonality condition

$$\int_D \varphi_n N^* \xi_m \, dV_p = 0 \quad \text{for} \quad n \neq m. \tag{33}$$

If

$$M = M^* \quad \text{and} \quad N = N^*$$

the operators are said to be "formally self-adjoint" [7]. If it turns out that the boundary conditions as well as the operators are the same, then

$$\xi_m = \varphi_m$$

and M and N are called self-adjoint operators.

Applying (33) to (12.a), (12.b), (22) and (23) we see that

$$c_{in} = \frac{\int_D v_i N^* \xi_n \, dV_p}{\int_D \varphi_n N^* \xi_n \, dV_p} , \tag{34}$$

$$e_{jn} = \frac{\int_D w_j N^* \xi_n \, dV_p}{\int_D \varphi_n N^* \xi_n \, dV_p} , \tag{35}$$

$$h_{1n} = \frac{\int_D H_1 N^* \xi_n \, dV_p}{\int_D \varphi_n N^* \xi_n \, dV_p} , \tag{36}$$

$$h_{2n} = \frac{\int_D H_2 N^* \xi_n \, dV_p}{\int_D \varphi_n N^* \xi_n \, dV_p} .$$

*Theorem 4.1, page 199, of Reference [7].

4. Discussion. Since the eigenvalue problems for the case of steady excitation problems, $F = F(P)$ and $G = G(P)$, are the same as those for time-dependent excitations $F = F(P, t)$ and $G = G(P, t)$, existing results can be easily transformed to the more general case of time-dependent inputs. The main difference between these cases is in the solution of the time equations, (16), (24) and (25). Examples for which the present method can be applied are:

Poisson equation: $\nabla^2 u + f(P, t) = 0,$

Wave equation: $\nabla^2 u = (1/c^2)\, \partial^2 u/\partial t^2,$

Heat equation: $\nabla^2 u = (1/\alpha^2)\, \partial u/\partial t,$

Telegraph equation: $\nabla^2 u = Tu,$

Plate or beam equations: $\nabla^4 u = - \ /s\, \partial^2 u/\partial t^2,$

Elastic ring equation: $\frac{\partial^2}{\partial \theta^2} (\partial^2(\ )/\partial \theta^2 + 1)^2 u = T(\partial^2(\ )/\partial \theta^2 - k)u.$

It is interesting to note that the result

$$u = \sum_n \varphi_n \psi_n + \sum_i v_i k_i + \sum_j w_j q_j$$

consists of two series of quasi-static solutions superimposed upon a homogeneous-type solution. This is analogous to "complete" solutions of ordinary differential equations in which homogeneous (or complementary) and particular (or "steady-state") solutions represent components of the total solution.

5. Acknowledgements. The author is grateful to Professor R. M. Rosenberg for calling his attention to Reference [4]. He would also like to thank the referee for suggesting certain changes in the original manuscript.

References

1. M. Phillips, *Solution de divers problèmes de Mécanique...*, J. Math. Purés et App., (2) **9**, 25–83 (1864)
2. R. D. Mindlin and L. E. Goodman, *Beam vibrations with time-dependent boundary conditions*, J. App. Mech., **17**, 377 (1950)
3. S. Timoshenko, *Vibration Problems in Engineering*, D. Van Nostrand, N. Y., 3rd Ed., 1955
4. J. G. Berry and P. M. Naghdi, *On the vibration of elastic bodies having time-dependent boundary conditions*, Quart. App. Math. **14**, 43 (1956)
5. T. S. Chow, Quart. App. Math., *On the solution of certain differential equations by characteristic function expansions*, **16**, 227 (1958)
6. I. U. Ojalvo, *Conduction with time-dependent heat sources and boundary conditions*, to appear in the Int. J. Heat & Mass Transfer
7. B. Friedman, *Principles and techniques of applied mathematics*, J. Wiley & Sons, N. Y., 1956