THE ATEB(H)-FUNCTIONS AND THEIR PROPERTIES*

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Summary. The *Ateb*(*h*)-functions are inversions of incomplete Beta-functions. They are the solutions of normal mode vibrations of certain nonlinear multi-degree-of-freedom systems just as the trigonometric functions yield the normal mode vibrations of the corresponding linear systems. Like elliptic functions, the *Ateb*(*h*)-functions depend on a parameter. The *Ateb*-functions reduce to trigonometric functions, and the *Atebh*-functions to hyperbolic functions when the parameter is 1. When the parameter is 2, the *Ateb*-functions become elliptic functions. A number of properties of the *Ateb*(*h*)-functions, such as identities, derivatives, integrals, differential equations satisfied by them, etc., are given.

Introduction. The equation

$$x^{\cdot\cdot} + cx \mid x \mid^{k-1} = 0, \tag{1}$$

where c and k are distinct, positive finite constants, has a certain intrinsic interest in dynamics because it is the equation of motion of a mass suspended from a nonlinear spring that resists being deflected with a force proportional to the k-th power of the deflection. Actually, it has much wider applicability [1] because the motion in principal modes of a certain class of nonlinear systems having many degrees of freedom also satisfies (1). This class is composed of a chain of n masses m_i in which each mass has a translational degree of freedom u_i in the direction of the chain. Each mass is connected with one, several, or all others by nonlinear springs, and the force which the spring connecting m_p with m_q exerts on m_p is

$$F_{pq} = -a_{pq}(u_p - u_q) \mid u_p - u_q \mid^{k-1}, \tag{2}$$

where the a_{pq} are constants, distinct or not. These systems have been called "homogeneous" because the potential function (which exists) is a homogeneous function of degree $k + 1$ in the u_i .

It is the purpose of this paper to discuss the integrals $x(t)$ of (1) under two sets of boundary conditions which are appropriate for the dynamical problems mentioned above. They are denoted, respectively, as Case (I) and Case (II) and are defined by

Cases (I) and (II): $\quad x^{\cdot} = 0 \quad \text{when} \quad x = X > 0;$

Case (I): $\quad x = 0 \quad \text{when} \quad t = 0;$

Case (II): $\quad x = X \quad \text{when} \quad t = 0.$

While Case (II) prescribes conventional initial conditions (magnitude and slope of x are prescribed at the same instant), this is not true for Case (I). Instead, in the latter, the displacement is prescribed at the initial instant while the velocity $x^{\cdot}$ is given at some other instant when the displacement is X. Therefore, the set of conditions for

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Case (I) harbors the *a priori* assumption that there exists a real t when $x = X$. One must then show *a posteriori* that an initial value of $x^\cdot$ can be found such that this assumption is verified.

The integrals. Before integrating (1), that equation will be transformed by means of

$$\tau = (c/n)^{1/2} X^{n-1} t, \qquad x = \xi X, \qquad n = (k+1)/2 \tag{3}$$

into what will be regarded as the canonical form. If the prime denotes differentiation with respect to τ, this form is

$$\xi'' + n\xi \mid \xi \mid^{2(n-1)} = 0, \tag{4}$$

with $\frac{1}{2} < n < \infty$, and the boundary conditions become

$$\begin{aligned} &\text{Cases (I) and (II):} \qquad && \xi' = 0 \quad \text{when} \quad \xi = 1; \\ &\text{Case (I):} && \xi = 0 \quad \text{when} \quad \tau = 0; \\ &\text{Case (II):} && \xi = 1 \quad \text{when} \quad \tau = 0. \end{aligned} \tag{5}$$

Evidently, (1) and (4) become linear for $k = n = 1$; therefore, they are generalizations of the linear case, and all results deduced from (4) and (5) must reduce to well-known formulas when $n = 1$.

An interesting feature of the transformations (3) is that, when $n \neq 1$, the transformed time τ is a function of the amplitude X while t is not. Since the transformed time stretches or shrinks with the amplitude, the results with respect to anisochronism will be different for (2) and (4).

A first integral of (4) is for both

$$\text{Cases (I) and (II):} \qquad \xi' = \pm(1 - \mid \xi \mid^{2n})^{1/2}, \tag{6}$$

and the second integrals are

$$\text{Case (I):} \qquad \tau = \pm \int_0^{0 \le \xi \le 1} (1 - \mid r \mid^{2n})^{-1/2}\, dr; \tag{7}$$

$$\begin{aligned} \text{Case (II):} \qquad \tau &= \pm \int_1^{0 \le \xi \le 1} (1 - \mid r \mid^{2n})^{-1/2}\, dr \\ &= \pm \left\{ \int_0^{0 \le \xi \le 1} (1 - \mid r \mid^{2n})^{-1/2}\, dr - \int_0^1 (1 - \mid r \mid^{2n})^{-1/2}\, dr \right\} \end{aligned} \tag{8}$$

The question [mentioned in connection with the discussion of the boundary conditions of Case (I)] whether a τ_0 exists such $\xi(\tau_0) = 1$ is simply one of the existence of (7) when $\xi = 1$. Quite obviously, this integral does exist and, in fact, its value is a complete Beta-function. Consequently, a $\xi'(0)$ can be found such that the first of (5) is satisfied; it is, in view of the second of (5) and of (6), $\xi'(0) = 1$.

The sign ambiguities in (7) and (8) are easily resolved. Since $\tau \geq 0$, the positive sign in (7) is the only admissible one, and since one can show readily for (8) that $\int_0^1 > \int_0^{0 \le \xi < 1}$, the negative sign must be chosen for Case (II).

It was noted by Miishkes [2] and others [3, 4, 5] that the transformation $r^{2n} = s$ transforms the second integral of (8) into

$$\int_0^1 (1 - \mid r \mid^{2n})^{-1/2}\, dr = \frac{1}{2n} \int_0^1 s^{(1-2n)/2n} (1 - s)^{-1/2}\, ds = \frac{1}{2n} B\left(\frac{1}{2n}, \frac{1}{2}\right)$$

where $B(1/2n, \frac{1}{2})$ is a complete Beta-function $B(p, q)$. By means of the same transformation, the integral in (7) [which is the same as the first in (8)] becomes

$$\int_0^{0\le\xi\le1} (1 - |r|^{2n})^{-1/2}\, dr = \frac{1}{2n} B_{\xi^*}\left(\frac{1}{2n}, \frac{1}{2}\right), \qquad \xi^* = \xi^{1/2n}$$

where $B_{\xi^*}(1/2n, \frac{1}{2})$ is the incomplete Beta-function $B_{\xi^*}(p, q)$, widely used in statistics and tabulated for a variety of values of p and q, and of its argument ξ^* in the interval $0 < \xi^* \le 1$ [6]. However, having been computed with the needs of the statistician in mind, the range and mesh on p, when $q = \frac{1}{2}$, is entirely inadequate for the dynamical problem under study here.

In the interval $0 \le \xi^* \le 1$, the incomplete Beta-function as well as its first derivative with respect to ξ^* are monotone functions of ξ^*. Therefore, when the upper limit of the integral is $\xi^* = 1$, one obtains the quarter period. Denoting the period of $\xi(\tau)$ by L, and that of $x(t)$ by T, one finds

$$\begin{aligned} L &= \frac{2}{n} B\left(\frac{1}{2n}, \frac{1}{2}\right), \\ \left(\frac{2c}{k+1}\right)^{1/2} X^{(k-1)/2} T &= \frac{4}{k+1} B\left(\frac{1}{2n}, \frac{1}{2}\right). \end{aligned} \tag{9}$$

Therefore, $\xi(\tau) = \xi(\tau + L)$ is isochronous while, in general, $x(t) = x(t + T)$ is anisochronous. In the linear case $k=n=1$, one finds the well-known results $L=2B(\frac{1}{2}, \frac{1}{2})=2\pi$, and $T = 2\pi/c^{1/2}$.

In summary, the periods are given by (9), and the solutions of the problem in hand are for

Case (I):
$$\tau = \frac{1}{2n} B_{\xi^*}\left(\frac{1}{2n}, \frac{1}{2}\right) \tag{10}$$

Case (II):
$$\tau = \frac{1}{2n}\left[B\left(\frac{1}{2n}, \frac{1}{2}\right) - B_{\xi^*}\left(\frac{1}{2n}, \frac{1}{2}\right)\right] \tag{11}$$

The inversions. The domain of single-valuedness of the functions $\tau(\xi)$ in (10) and (11) certainly cannot exceed a quarter period $L/4$ while the inversions $\xi(\tau)$ are single-valued on any interval. For this as well as other obvious reasons, these functions will now be inverted. On self-evident grounds, the inversions have been named *Ateb*-functions. In a general way, these inversions will be modeled on that of the elliptic integral of the first kind. However, the transformations used, and the results obtained differ from the elliptic functions.

We examine first the inversion of (7)

$$u_1 = \int_0^{\xi_1} (1 - |r|^{2n})^{-1/2}\, dr = F_1(\xi_1). \tag{12}$$

We introduce the transformations

$$\xi_1 = \sin^{1/n}\varphi_1, \qquad r = \sin^{1/n}\theta \tag{13}$$

with the understanding that $\sin^{1/n}\varphi_1$ has the same sign as $\sin\varphi_1$. This would be formally insured by using, instead of (13), $\xi_1 = \sin\varphi_1 \,|\sin\varphi_1|^{(1-n)/n}$ and a similar expression for r. However, we can avoid this detail by means of the following *rule*:

Exponents n and $1/n$ behave like odd integers.
Under (13), the integral (12) becomes

$$nu_1 = \int_0^{0 \leq \varphi_1 \leq \pi/2} \sin^{(1-n)/n} \theta \, d\theta, \tag{14}$$

where the principal value of the first of (13) was used to establish the upper limit.

The exponent

$$(1 - n)/n = h(n), \qquad (1/2 < n < \infty)$$

in (14) is bounded. In fact, $h(\frac{1}{2}) = 1$; $h(1) = 0$; $h(n) < 0$ when $n > 1$; $h(n) \to -1$ as $n \to \infty$; $h'(n) \neq 0$; $h'(1) = -1$; $h'(n) \to 0$ as $n \to \infty$. Therefore, $h(n)$ looks as shown in Figure 1.

We now define, similarly to elliptic function theory the *Ateb*-function

$$\varphi_1 = \operatorname{amp} nu_1 \tag{15}$$

as the amplitude of nu_1. (To distinguish it from the corresponding operator in the theory of elliptic functions, the letter "p" has been retained in "amp.") Then, the substitution of (15) in the first of (13) defines the *Ateb*-function

$$\xi_1 = \sin^{1/n} (\operatorname{amp} nu_1) \equiv \operatorname{sam} (nu_1) \tag{15}$$

as the inversion of (12). Evidently, it corresponds to the elliptic *sn* function. Its name was chosen because an *Ateb*-function corresponding to the elliptic *cn* function also exists, but a counterpart to the *dn* function does not.

Next we consider the inversion of (8)

$$u_2 = -\int_1^{0 \leq \xi_2 \leq 1} (1 - |r|^{2n})^{-1/2} \, dr = F_2(\xi_2). \tag{16}$$

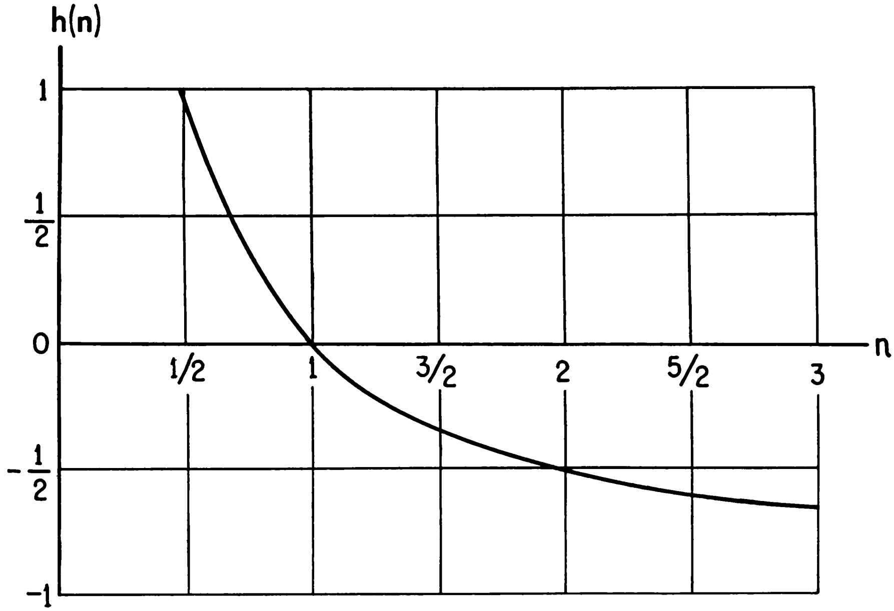

FIG. 1

With the transformations

$$\xi_2 = \cos^{1/n} \varphi_2 , \qquad r = \cos^{1/n} \theta \tag{17}$$

and the rule given above, (16) becomes

$$nu_2 = \int_0^{0 \le \varphi_2 \le \pi/2} \cos^{(1-n)/n} \theta \, d\theta. \tag{18}$$

Then,

$$\varphi_2 = \operatorname{amp} nu_2 \tag{19}$$

substituted into the first of (17) yields the *Ateb*-function

$$\xi_2 = \cos^{1/n} (\operatorname{amp} nu_2) \equiv \operatorname{cam} (nu_2) \tag{20}$$

as the inversion of (16).

In summary, the solutions of the dynamical problems are

Case (I): $\xi = \operatorname{sam} (n\tau);$

Case (II): $\xi = \operatorname{cam} (n\tau);$

and the *Ateb*-functions amp nu, sam (nu), and cam (nu) are all inversions of an incomplete Beta-function.

Some properties of the Ateb-functions. The Ateb-functions

$$\varphi_{1,2} = \operatorname{amp} nu_{1,2} = f_{1,2}(u)$$

are the inversions of

$$u_{1,2} = \int_0^{\varphi_{1,2}} [v_{1,2}(\theta)]^{(1-n)/n} \, d\theta = g_{1,2}(\theta)$$

where $v_1(\theta) = \sin \theta$ and $v_2(\theta) = \cos \theta$. Under the above rule, $(1 - n)/n$ behaves like an even exponent so that $[v_{1,2}(\theta)]^{(1-n)/n}$ are both even in θ; hence, their integrals are odd in φ, or

$$\operatorname{amp} - nu_{1,2} = -\operatorname{amp} nu_{1,2} . \tag{21}$$

Evidently, amp $nu_{1,2}$ is continuous at the origin because one has from (14) and (18)

$$\operatorname{amp} 0 = 0. \tag{22}$$

When $\xi_1 = 1, \varphi_1 = \pi/2$ because of (13), and when $\xi_2 = 0, \varphi_2 = \pi/2$ because of (17). Moreover, when $\xi_1 = 1$ and when $\xi_2 = 0$ the integrals (12) and (16) become identical. Therefore, when $u_{1,2}$ are regarded, respectively, as functions of $\xi_{1,2}$, and the notation

$$u_1(1) = u_1^* , \qquad u_2(0) = \mu_2^* \tag{23}$$

is used, one finds

$$u_1^* = u_2^* = u^* = \frac{1}{2n} B\left(\frac{1}{2n} , \frac{1}{2}\right) \tag{24}$$

and, in consequence,

$$\text{amp } nu^*_{1,2} = \pi/2. \tag{25}$$

Finally, when $n = 1$ (the linear case), it is seen from (14) and (18) that $u_{1,2} = \varphi_{1,2}$, or

$$\text{amp } u_{1,2} = u_{1,2}\,; \qquad (n = 1). \tag{26}$$

Therefore, both functions $\varphi_{1,2} = \text{amp } nu_{1,2}$ are odd functions of, and vanish with, their arguments; both take on the value $\pi/2$ when $u_{1,2} = u^* = 1/2nB(1/2n, \frac{1}{2})$; and both reduce to identical straight lines of slope unity when $n = 1$. However, in spite of these common properties, these functions are not identical. This is seen from their derivatives which are readily found by direct differentiation as

$$\begin{aligned} \frac{d}{du_1}(\text{amp } nu_1) &= n \text{ sam}^{n-1}(nu_1), \\ \frac{d}{du_2}(\text{amp } nu_2) &= n \text{ cam}^{n-1}(nu_2). \end{aligned} \tag{27}$$

Special values of these derivatives are

$$\left.\begin{aligned} \frac{d}{du_1}(\text{amp } 0) &= 0 \quad \text{when} \quad n > 1, \\ &= 1 \quad \text{when} \quad n = 1, \\ &= \infty \quad \text{when} \quad n < 1; \\ \frac{d}{du_1}(\text{amp } nu_1^*) &= n; \end{aligned}\right\} \tag{28}$$

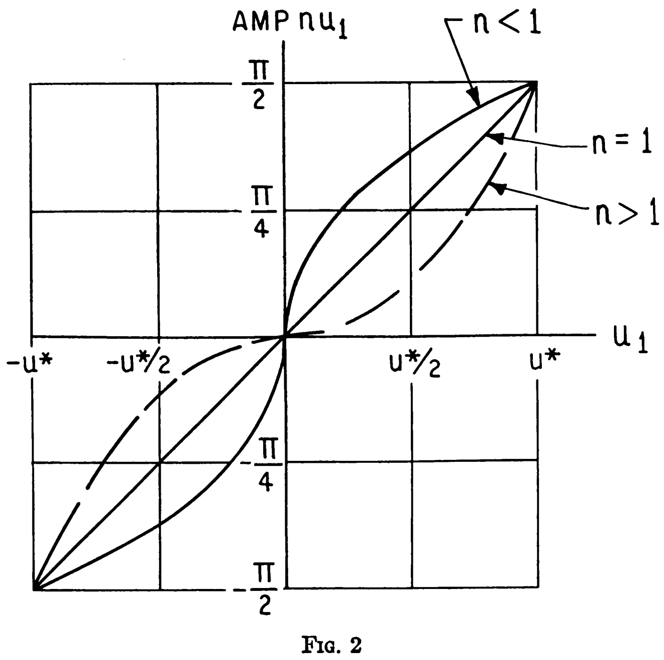

FIG. 2

$$\left.\begin{aligned}\frac{d}{du_2}(\text{amp } 0) &= n;\\ \frac{d}{du_2}(\text{amp } nu^*{}_2) &= 0 \quad \text{when} \quad n > 1,\\ &= 1 \quad \text{when} \quad n = 1,\\ &= \infty \quad \text{when} \quad n < 1.\end{aligned}\right\} \tag{29}$$

The functions amp nu_1 and amp nu_2 are shown, respectively, in Figures 2 and 3.

Combining curves of Figures 2 and 3, it is seen that a loop consisting of the positive branches of amp nu_1 and amp nu_2 (for the same value of n) has point symmetry with respect to the point ($u = u^*/2$, amp $nu = \pi/4$). Such a loop is shown in Figure 4. From it, one sees that

$$\text{amp } n(u^* \pm u_{1,2}) = \text{amp } nu^* \pm \text{amp } nu_{1,2} \tag{30}$$

where equal values of n, equal subscripts, and corresponding signs must be used. Since amp $nu^* = \pi/2$, one has the relation, familiar from elliptic function theory,

$$\text{amp } n(pu^* \pm u_{1,2}) = p\,\frac{\pi}{2} \pm \text{amp } nu_{1,2} \tag{31}$$

where p is a positive integer.

The remaining *Ateb*-functions are

$$\xi_1 = \text{sam } (nu_1)$$

$$\xi_2 = \text{cam } (nu_2)$$

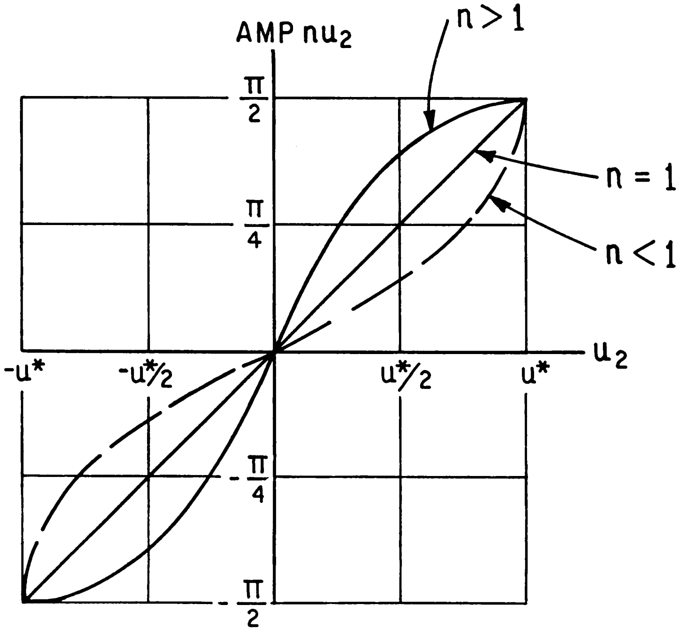

FIG. 3

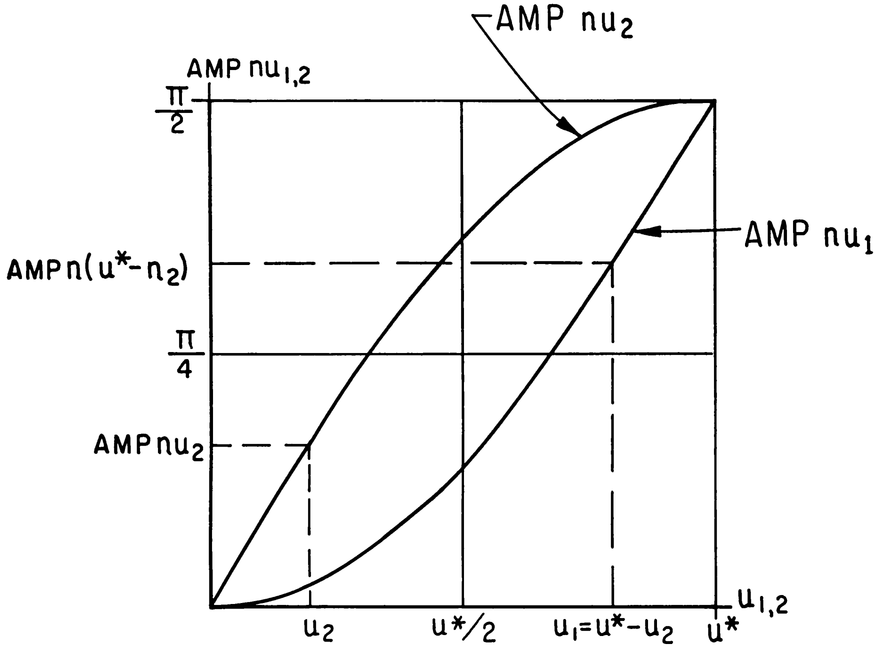

FIG. 4

One can readily show that they are periodic; this follows the observation that for both (14) and (18)

$$\int_0^{\varphi+2\pi} = \int_0^{\varphi} + \int_{\varphi}^{\varphi+2\pi} = \int_0^{\varphi} + \int_0^{2\pi} = \int_0^{\varphi}$$

and from (15), (19), and (25). In fact, one finds

$$\begin{aligned} \text{sam}\ (nu_1) &= \text{sam}\ (nu_1 + 4nu^*), \\ \text{cam}\ (nu_2) &= \text{cam}\ (nu_2 + 4nu^*). \end{aligned} \tag{32}$$

It follows that the period of these functions is $L = 4u^*$. This result agrees, as it must, with (9) and (24).

In view of the defining equations (15) and (20), of the fact that the amp functions are odd, and that the exponent $1/n$ behaves like an odd integer, one has, as expected,

$$\begin{aligned} \text{sam}\ (-nu_1) &= -\text{sam}\ (nu_1), \\ \text{cam}\ (-nu_2) &= \text{cam}\ (nu_2). \end{aligned} \tag{33}$$

Finally, combining the defining equations with (26)

$$\left.\begin{aligned} \text{sam}\ (u_1) &= \sin u_1 = \sin u_2 \\ \text{cam}\ (u_2) &= \cos u_2 = \cos u_1 \end{aligned}\right\} n = 1 \tag{34}$$

It is not difficult to derive a number of identities for the sam and cam functions which correspond to the well-known trigonometric identities. However, they are more complicated than the latter because they connect these functions for those values $u_1 = u_1^0$, and $u_2 = u_2^0$ for which

$$\operatorname{amp} nu_1^0 = \operatorname{amp} nu_2^0 . \tag{35}$$

Direct application of (31), and putting $p = 1$, results in

$$\operatorname{amp} n(u^* + u_1^0) + \operatorname{amp} n(u^* - u_2^0) = \pi. \tag{36}$$

This is a relation between those values of the arguments u_1 and u_2 for which their amplitude functions are equal. When it is satisfied, one can show readily that

$$\operatorname{sam}^{2n} (nu_1^0) + \operatorname{cam}^{2n} (nu_1^0) = 1. \tag{37}$$

The first two derivatives of these *Ateb*-functions may be found by direct differentiation; they are

$$\begin{aligned}
\frac{d}{du_1^0} [\operatorname{sam} (nu_1^0)] &= \operatorname{cam}^n (nu_2^0), \\
\frac{d}{du_2^0} [\operatorname{cam} (nu_2^0)] &= -\operatorname{sam}^n (nu_1^0), \\
\frac{d^2}{du_1^2} [\operatorname{sam} (nu_1)] &= -n \operatorname{sam}^{2n-1} (nu_1), \\
\frac{d^2}{du_2^2} [\operatorname{cam} (nu_2)] &= -n \operatorname{cam}^{2n-1} (nu_2).
\end{aligned} \tag{38}$$

An interesting result is found for the integrals over one-quarter period; i.e., between 0 and $u^* = 1/2nB(1/2n, \frac{1}{2})$. For instance

$$J = \int_0^{u^*} \operatorname{sam} (nu_1)\, du_1 \tag{39}$$

can be readily converted into, and evaluated as [7]

$$J = \frac{1}{n} \int_0^{\pi/2} \sin^{(2-n)/n} \varphi_1 \, d\varphi_1 = \frac{1}{2n} B(1/n, 1/2) \tag{40}$$

provided the exponent $(2 - n)/n = m(n) > -1$. Evidently, (40) is valid in the entire range $\frac{1}{2} \leq n < \infty$ because $m(\frac{1}{2}) = 3$, and $m(n) \to -1$ as $n \to \infty$. The value of the integral (39) is related to its upper limit in an obvious manner.

When $n = 1$, all relations derived here reduce to those familiar from trigonometric functions. All of these reductions can be made by inspection, and with the aid of $B(\frac{1}{2}, \frac{1}{2}) = \pi$, $B(1, \frac{1}{2}) = 2$.

When $n = 2$ (equivalent to $k = 3$), another well-known special case arises. In that case, (12), for instance, takes on the form

$$u_1 = \int_0^{\xi_1} [Q(r)]^{-1/2}\, dr$$

where $Q(r)$ is a fourth degree polynomial. Such an integral always reduces to an elliptic integral of the first kind [8], and hence its inversion is an elliptic function. In fact, for $n = 2$ one has from (7)

$$u_1 = \frac{1}{\sqrt{2}} \int [(1 - r^2)(1 + r^2)]^{-1/2}\, dr \tag{41}$$

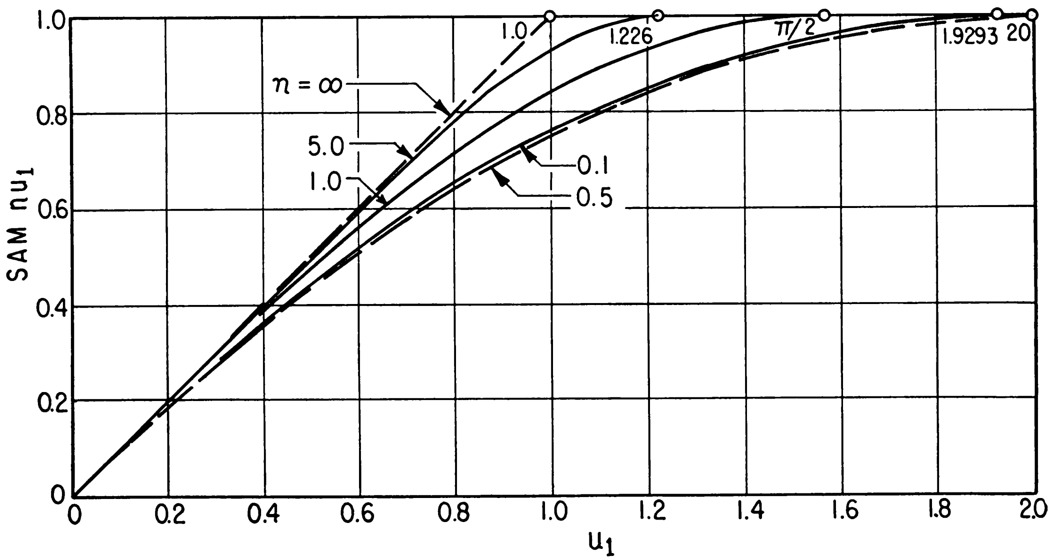

FIG. 5

and with $r = \cos\theta$, this reduces to

$$u_1 = -\frac{1}{\sqrt{2}} \int [1 - (1/2) \sin^2 \theta]^{-1/2} \, d\theta \tag{42}$$

Therefore, when $n = 2$, the inversion is an elliptic function in which the parameter has the values $2^{-1/2}$.

Graphs of sam (nu_1) and cam (nu_2) are shown in Figures 5 and 6 for several values of n. From these illustrations, a curious lack of symmetry between the sam and cam

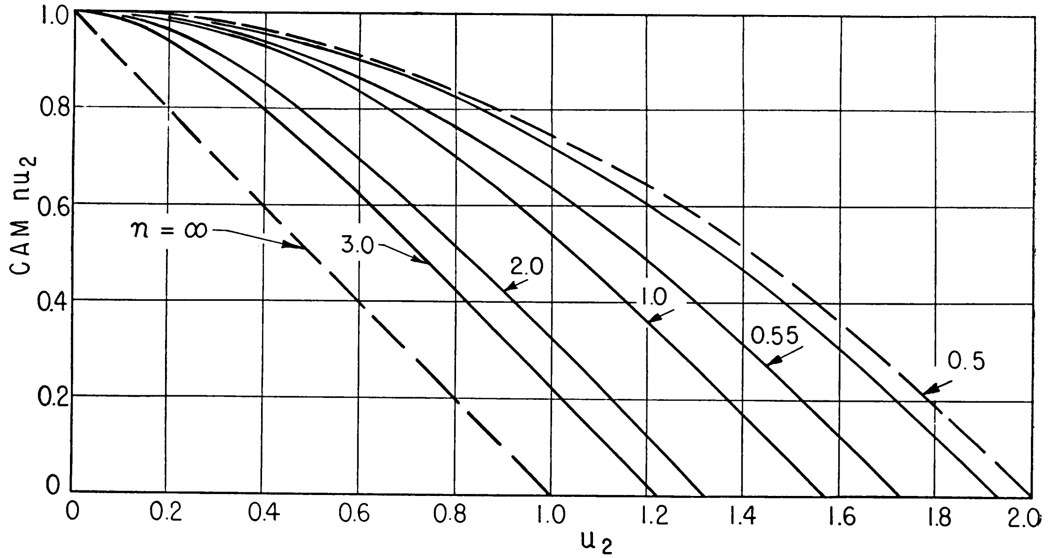

FIG. 6

functions becomes evident. It is quite different from the "dissymmetry" between corresponding elliptic *sn* and *cn* functions because, in the case of the elliptic functions, the actual wave shapes between corresponding *sn* and *cn* differ; this is not the case here. Instead, the sam function occupies a very much smaller portion of the rectangle $0 \leq \operatorname{sam}(nu_1) \leq 1, 0 \leq u_1 \leq 2$ than the cam function does in a corresponding rectangle. Basically, this lack of symmetry is due to that between amp nu_1 and amp nu_2.

Extensive tables of the sam and cam functions are being computed. It is hoped that they will be published before long.

Finally, it should be mentioned that the trigonometric *Ateb*-functions can be readily generalized to hyperbolic *Ateb*-functions (*Atebh*-functions). The latter arise when the condition that the constant c in (1) be positive finite is replaced by $-\infty < c < 0$. Then, the same definitions and transformations used to define the *Ateb*-functions now lead to the functions samh and camh which are connected with the trigonometric functions by

$$\begin{aligned} \operatorname{samh}(inu) &= i \operatorname{sam}(nu), \\ \operatorname{camh}(inu) &= \operatorname{cam}(nu). \end{aligned} \tag{43}$$

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