

## — NOTES —

### A NOTE ON EXTRAPOLATION PROCEDURES FOR SOLVING LINEAR SYSTEMS\*

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**1. Introduction.** The purpose of this note is two-fold. First, we point out a difference between the extrapolation procedure used in a previous paper [1] (hereafter referred to as I) and the more standard procedure [2] which is in general use†. Then we point out how to modify the extrapolation procedure used in this method (referred to hereafter as Method Ia to distinguish it from Method I discussed in I as a special case) so that it is equivalent to the more standard procedure.

We discuss the same problem as was discussed in I and we use the same notational conventions as were used there. For an introduction to the problem we refer the reader to I. The matrix form of our difference equation is

$$B\theta + s = 0. \quad (1)$$

The  $B$  matrix is broken up into lower diagonal  $L$ , upper diagonal  $U$  and diagonal  $D$  parts.

$$B = L + U + D \quad (2)$$

Then (1) can be written,

$$(L + U + D)\theta = -s \quad (3)$$

or

$$(L + D)\theta = -s - U\theta \quad (4)$$

The unextrapolated case ( $\omega = 1$ ) of Method Ia reduces for  $k = 0$  to

$$(L + D)\theta^{(n+1)} = -s - U\theta^{(n)} \quad (5)$$

This is known [2] as the Liebmann method. Extrapolated Method Ia reduces to

$$(L + D)\theta^{(n+1)} = \omega[-s - U\theta^{(n)}] + (1 - \omega)(L + D)\theta^{(n)} \quad (6)$$

for  $k = 0$ . However the commonly known [2] extrapolated Liebmann method is obtained by first transposing the  $L\theta^{(n+1)}$  term of (5) to the right hand side and then extrapolating.

$$D\theta^{(n+1)} = \omega[-s - U\theta^{(n)} - L\theta^{(n+1)}] + (1 - \omega)D\theta^{(n)} \quad (7)$$

This is sometimes written in the form

$$D\theta^{(n+1)} = D\theta^{(n)} - \omega[s + U\theta^{(n)} + L\theta^{(n+1)} + D\theta^{(n)}] \quad (8)$$

It was implied in I that Method Ia reduces for  $k = 0$  to the extrapolated Liebmann method. However, this is not the case since it reduces to (6) whereas the extrapolated

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Liebmann method is given by (7) or (8). We indicate below how to modify Method Ia so that it reduces to (7) for  $k = 0$ .

**2. Modification of the Previous Method.** In I we rewrote (3) as

$$(L + kU + D)\theta = -s - (1 - k)U\theta \quad (9)$$

Now, we choose to transpose part of the  $L\theta$  term as well.

$$(\alpha L + kU + D)\theta = -s - (1 - k)U\theta - (1 - \alpha)L\theta \quad (10)$$

Thus, we write

$$C\theta = \mathbf{d} \quad (11)$$

where

$$C = \alpha L + kU + D \quad (12)$$

and

$$\mathbf{d} = -s - (1 - k)U\theta - (1 - \alpha)L\theta \quad (13)$$

From this point on the algebra goes just as it did in I with the reservation that  $C$  and  $\mathbf{d}$  are defined in a slightly different way. The boundary conditions are included in the precise manner as in I. The final computational formulas corresponding to those of Section 3 of I modified to include the  $\alpha$  transposition of (10) above are summarized as follows.

$$W_i^{i-N} = \alpha B_i^{i-N} \Gamma_{i-N} \quad (14)$$

$$W_i^{i-1} = \alpha B_i^{i-1} \Gamma_{i-1} \quad (15)$$

$$W_i^i = B_i^i - W_i^{i-N} V_{i-N}^i - W_i^{i-1} V_{i-1}^i \quad (16)$$

$$V_i^{i+N} = \frac{k B_i^{i+N}}{W_i^i} \Gamma_{i+N} \quad (17)$$

$$V_i^{i+1} = \frac{k B_i^{i+1}}{W_i^i} \Gamma_{i+1} \quad (18)$$

$$\begin{aligned} h_i = & -s_i + [W_i^{i-N} V_{i-N}^{i-N+1} \theta_{i-N+1}^{(n)} \Gamma_{i-N+1} + W_i^{i-1} V_{i-1}^{i+N-1} \theta_{i+N-1}^{(n)} \Gamma_{i+N-1}] \\ & - (1 - k)[B_i^{i+N} \theta_{i+N}^{(n)} \Gamma_{i+N} + B_i^{i+1} \theta_{i+1}^{(n)} \Gamma_{i+1}] \\ & - (1 - \alpha)[B_i^{i-N} \theta_{i-N}^{(n)} \Gamma_{i-N} + B_i^{i-1} \theta_{i-1}^{(n)} \Gamma_{i-1}] \\ & - [B_i^{i-N} \theta_{i-N}^{(b)} \Delta_{i-N} + B_i^{i-1} \theta_{i-1}^{(b)} \Delta_{i-1} \\ & + B_i^{i+1} \theta_{i+1}^{(b)} \Delta_{i+1} + B_i^{i+N} \theta_{i+N}^{(b)} \Delta_{i+N}] \end{aligned} \quad (19)$$

$$g_i = \frac{h_i - W_i^{i-N} g_{i-N} \Gamma_{i-N} - W_i^{i-1} g_{i-1} \Gamma_{i-1}}{W_i^i} \quad (20)$$

$$\theta_i^{(n+1)} = \omega [g_i - V_i^{i+N} \theta_{i+N}^{(n+1)} \Gamma_{i+N} - V_i^{i+1} \theta_{i+1}^{(n+1)} \Gamma_{i+1}] + (1 - \omega) \theta_i^{(n)} \quad (21)$$

The only differences between the above results and those of I are the  $\alpha$  factor in (14) and (15) and the  $(1 - \alpha)$  term in (19).

The only modification in the convergence condition given in Section 4 of I is that  $K$  is now given by

$$K = V^{-1} W^{-1} [H - (1 - k)U - (1 - \alpha)L] \omega + (1 - \omega) \quad (22)$$

To obtain extrapolated Liebmann as defined by (7) we simply set  $k = 0$  and  $\alpha = \omega$ . The generalization which is analogous to Method Ia involves setting  $\alpha = \omega$  and allowing  $k$  to vary. We call this case Method Ib. The difference between Method Ib and Method Ia is that we set  $\alpha = \omega$  for the former and  $\alpha = 1$  for the latter. In Figure 1 we have plotted a family of curves for Method Ib for the same problem considered in I for various values of  $k$ . The curve for  $k = 0$  corresponds to extrapolated Liebmann<sup>2</sup> or successive over-relaxation. An analogous family of curves for Method Ia can be found in I. In I (as illustrated by the graphs given there) we pointed out the great increase in convergence rates for higher values of  $k$  over the  $k = 0$  case. However, we see in Figure 1 here that higher values of  $k$  give very little increase in convergence rate over the  $k = 0$  case. In view of the fact that in the  $k = 0$  (extrapolated Liebmann) case the computational formulas become greatly simplified, it seems that one cannot obtain any practical improvement by taking higher values of  $k$ . Thus, we conclude that the partial transposition of the  $U$  matrix as done in Methods Ia and Ib cannot be expected to yield a practical improvement over extrapolated Liebmann.

There remains, however, an interesting case which we call Method Ic. If instead of taking  $\alpha = 1$  in Method Ia, we take  $\alpha = 1.6$  (the best  $\omega$  for Method Ib), and hold it at

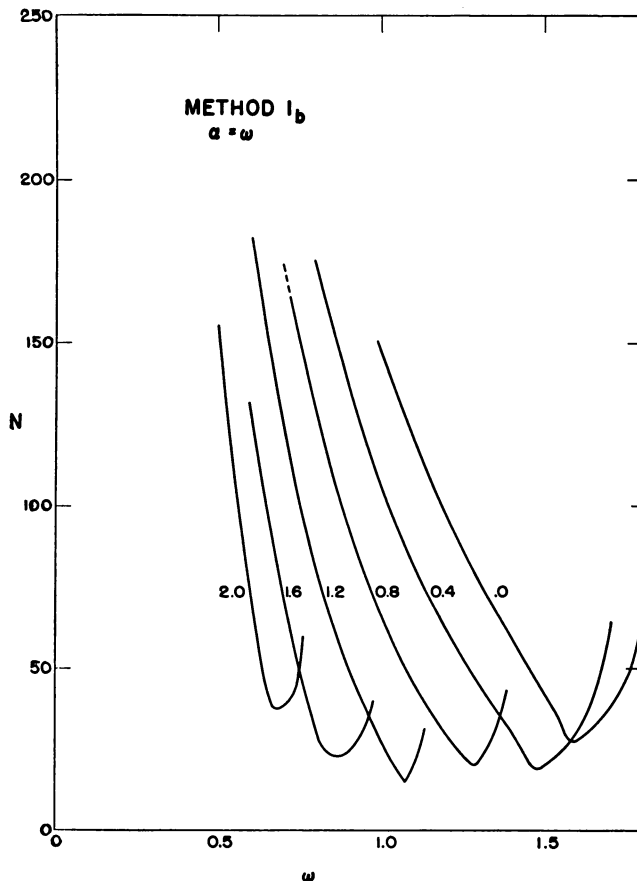


FIG. 1. The number of iterations  $N$  is plotted against the extrapolation parameter  $\omega$  for various values of the parameter  $k$  using Method Ib. The value of  $k$  for each curve is indicated in the graph.

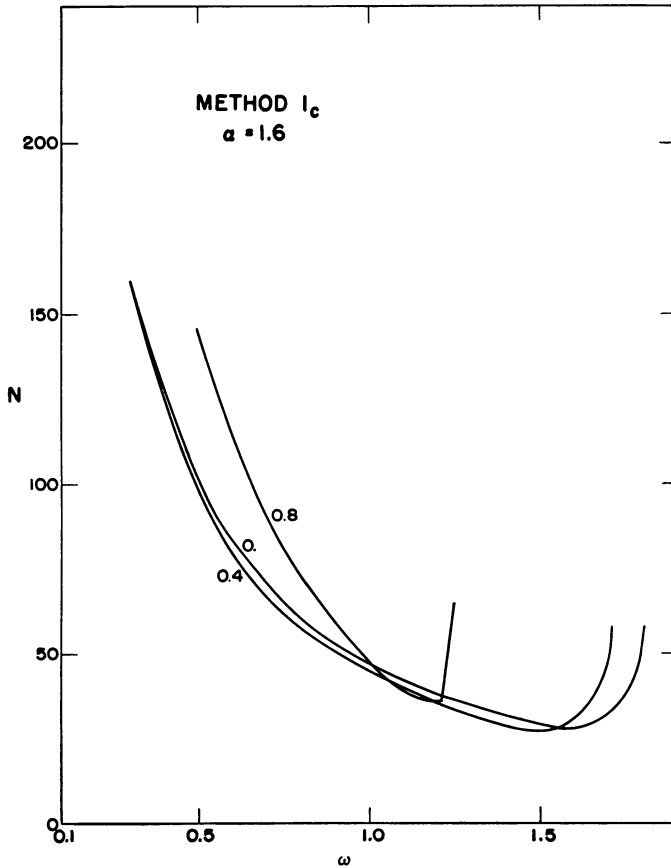


FIG. 2. The number of iterations  $N$  is plotted against the extrapolation parameter  $\omega$  for various values of the parameter  $k$  using Method  $I_c$ . The value of  $k$  for each curve is indicated in the graph.

1.6 instead of letting it get small with  $\omega$ , then the convergence rates stay good for a much greater range of values of  $\omega$  on the low side and things are not made much worse on the high side. A family of curves for Method  $I_c$  is given in Figure 2. Clearly, the best case here is  $k = 0$ . The computational formulas for  $\alpha \neq \omega$  are still more complicated than for  $\alpha = \omega$  and hence one obtains a greater latitude for the possible values of  $\omega$  at the price of a more complicated computation. Another difficulty is that there is no systematic way of knowing what  $\alpha$  to use for a general problem. Thus, it is difficult in this case as well as in the above cases to see how to get a general practical improvement over extrapolated Liebmann.

#### REFERENCES

1. T. A. Oliphant, *An Extrapolation Procedure for Solving Linear Systems*, Quart. Appl. Math., to be published.
2. S. P. Frankel, *Convergence Rates of Iterative Treatment of Partial Differential Equations*, Math. Tables Aids Computation 4, 65-75 (1950).