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UPPER AND LOWER BOUNDS FOR THE SOLUTION OF A MELTING PROBLEM*

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Abstract. A slab, melting under an arbitrary heat input on one face and insulated on the other is studied. Variable material properties are taken into account. After preliminary general considerations, it is shown that the solution to the stated problem is unique. It is then proved that higher rates of melting and higher temperatures will result from higher heating rates; from this intuitively plausible result upper and lower bounds for the solution are easily constructed. An example is worked out in detail.

1. Introduction. This paper presents several theorems concerning the problem of heat conduction in melting (or solidifying) slabs. The principal results are the establishment of a uniqueness theorem, and the development of a simple method for the construction of upper and lower bounds for the rate of melting and for the temperature. The problem considered is that of a slab, heated in an arbitrarily prescribed manner on one (moving) face and insulated on the other; the thermal properties are allowed to be temperature dependent. This problem may be taken to represent the case of a melting slab, with melted portion instantaneously removed.[1]

Section 2 of this paper presents two basic theorems concerning the solutions of a general type of parabolic differential equation with variable coefficients for arbitrary domains in the xt-plane. The problem of the melting slab is formulated mathematically in Section 3, and the upper and lower bounds mentioned earlier are constructed in Section 4. The discussion of Section 5 indicates how these bounds can be applied in the practical solution of a melting problem, and a detailed example is presented in Section 6.

In the proofs which follow frequent use is made of *Picone's Theorem* [6], a restricted statement of which is given here for convenience of reference:

Consider a domain D with boundary B in the $(n + 1)$-dimensional space of points P with coordinates $x_1, x_2, \cdots x_n, t$. Let B_{-t} be that part of B which includes all points P such that (a) the interior normal to the boundary exists and is directed in the negative t direction, and (b) each point P is an interior point of B_{-t}.[2] Let $u(P)$ be a solution of the

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[1]The solution of this problem is a good approximation [1], at short times, to that of the problem in which none of the melted material is removed; it may thus be a good approximation also for intermediate rates of ablation. The present problem also provides a good approximation to the problem of aerodynamic ablation, provided the Prandtl Number is small compared to unity [2, 3, 4, 5].

[2]For example, for the domain enclosed by the boundary $ABEE'B'A'A$ in Fig. 2, the portion B_{-t} is EE'.

parabolic differential equation (in class $C^{(2)}$):

$$\sum_{i,j}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu - \frac{\partial u}{\partial t} = 0, \tag{1}$$

where $a_{ij}(P) = a_{ji}(P)$, $b_i(P)$ and $c(P)$ are real finite continuous functions of P, and where the quadratic form $\sum_{ij}^{n} a_{ij}x_i x_j$ is positive-definite (singular or non-singular). Then Picone's Theorem states that:

(a) if $u \geq 0$ on $B - B_{-t}$, then $u \geq 0$ throughout D, and if $u \leq 0$ on $B - B_{-t}$, then $u \leq 0$ throughout D;
(b) if $u \equiv 0$ on $B - B_{-t}$, then $u \equiv 0$ throughout D;
(c) if u is prescribed throughout $B - B_{-t}$, then u is uniquely determined throughout D;
(d) the maximum value of $|u|$ occurs on $B - B_{-t}$; if $c \equiv 0$, both the maximum and the minimum values of u occur on $B - B_{-t}$.

A proof of Picone's theorem for the one-dimensional case needed below and for the case in which the coefficients a_{ij}, b_i and c are functions of u is given in [10]; all the results of this paper hold for this case.

2. Basic theorems. *Theorem I.* Let $u(P)$ be a solution, in class $C^{(2)}$, of the following special case of Eq. (1):

$$a(P) \frac{\partial^2 u}{\partial x^2} + b(P) \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0; \qquad a \geq 0 \tag{2}$$

at all points $P(x, t)$ of a domain D in the xt-plane, and let[3]

$$\frac{\partial u}{\partial n_x} = f(P), \qquad P \text{ on } B - B_{-t}, \tag{2a}$$

$$u = 0 \text{ at some point } P_1 \text{ on } B - B_{-t}, \tag{2b}$$

where n_x denotes the direction of the component parallel to the x-axis of the interior normal to the boundary B. Then,

(a) if $f(P) \equiv 0$, $u \equiv 0$ throughout D;
(b) if $f(P)$ is prescribed throughout $B - B_{-t}$, then u is uniquely determined throughout D.

Proof. With $v \equiv (\partial u/\partial x)$, differentiation of (2) gives (differentiability of a and b being assumed):

$$a \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial a}{\partial x} + b\right) \frac{\partial v}{\partial x} + \frac{\partial b}{\partial x} v - \frac{\partial v}{\partial t} = 0 \tag{3}$$

which is itself a special case of (1). If $f \equiv 0$, then $v \equiv 0$ on $B - B_{-t}$, and hence throughout D, by Picone's Theorem; this implies $u = u(t)$, and use of (2) and (2b) then gives $u \equiv 0$. This proves statement (*a*); statement (b) follows as a corollary.

Theorem II. Consider a simply connected domain D in the xt-plane, whose boundary is (Fig. 1) a segment of the line $x = x_0$, and a line defined by a continuous single-valued

[3]The extension to the case $u \neq 0$ at P_1 is trivial.

function $x = F(t)$ satisfying Lipschitz conditions for $t_1 < t < t_2$ and intersecting the line $x = x_0$ at two distinct points $P_1 : (x_0, t_1)$ and $P_2 : (x_0, t_2)$. Let $u(P)$ be the solution of Eq. (2) in D with

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= f(t) \quad \text{on} \quad x = F(t), \\ \frac{\partial u}{\partial x} &= 0 \quad \text{on} \quad x = x_0, \end{aligned}\right\} \text{for} \quad t_1 \le t \le t_2, \tag{4}$$

$$u = 0 \quad \text{at} \quad P_1 .$$

Then (a) if $f(t) \le 0$, $u \ge 0$ throughout D, and (b) if $f(t) \ge 0$, $u \le 0$ throughout D.

Proof. Extend the solution by reflection about $x = x_0$, so that (Fig. 1):

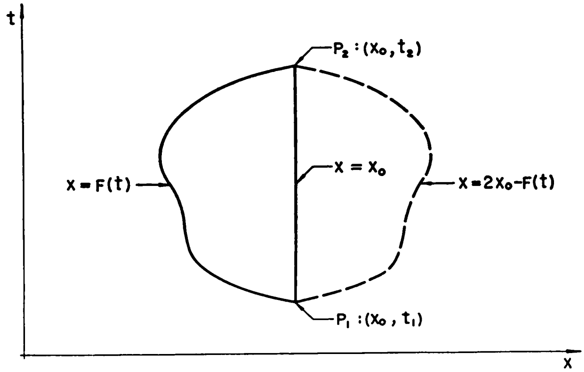

FIG. 1.

$$u(x, t) = u(2x_0 - x, t); \qquad F(t) \le x \le x_0 \tag{4a}$$

and so that u is now defined throughout the domain D_1 bounded by $x = F(t)$ and by $x = 2x_0 - F(t)$. Because of Theorem I, only the case $f(t) \not\equiv 0$ need be examined, and no loss of generality is therefore involved in setting, for case (a),

$$\begin{aligned} f(t) &= 0 \qquad t_1 \le t \le t', \\ f(t) &< 0 \qquad t' \le t \le t' + \delta, \qquad \delta > 0. \end{aligned} \tag{5}$$

Then $u \equiv 0$ for $t_1 \le t \le t'$, and a number δ', $0 < \delta' \le \delta$, clearly exists such that

$$u(P) > 0 \quad \text{on} \quad x = F(t), \qquad t' < t \le t' + \delta'. \tag{5a}$$

For, if this were not so, Picone's Theorem (applied to the domain D_1' consisting of the portion of D_1 within $t = t'$ and $t = t' + \delta'$) would require $u \le 0$ throughout D_1, with the minimum of u occurring on $x = F(t)$, and this contradicts the second of Eqs. (5). This proves that u is initially $(t = t' +)$ positive; but then it follows that u can never be negative, because, if it were, a later time $t'' > t' + \delta'$ would have to exist at which

$$\left.\begin{aligned} u &= 0, \qquad t = t'', \\ u &< 0, \qquad t'' < t \le t'' + \delta'', \qquad \delta'' > 0, \\ \frac{Du}{Dt} &< 0, \qquad t'' < t \le t'' + \delta''', \qquad 0 < \delta''' < \delta'', \end{aligned}\right\} \text{on} \quad x = F(t), \tag{6}$$

where Du/Dt stands for the total derivative of $u(x, t)$ along $x = F(t)$. Now, $f(t)$ cannot be negative in $t'' < t < t'' + \delta''$, since this would imply a minimum of u off the boundary; on the other hand it cannot be zero within $t'' < t < t'' + \delta'''$ because then the last of Eqs. (6) would reduce to $(\partial u/\partial t) < 0$ and hence Eq. (2) would require $(\partial^2 u/\partial x^2) < 0$, which again implies an interior minimum. Hence a non-vanishing interval exists within which a negative u is not compatible with $f \leq 0$; therefore $u \geq 0$ and part (a) of the theorem is proved. The proof of part (b) is entirely analogous.

3. Statement of the melting problem. Consider a slab, initially (i.e. at $t = 0$) at zero temperature and occupying the region $0 < x < L$, and insulated at $x = L$. An arbitrarily prescribed heat input $Q(t)$ is applied at $x = 0$, so that the temperature in the slab rises and at $x = 0$ reaches the melting temperature T_m at the time $t = t_m$. Melting continues to take place thereafter, and it is assumed that a portion of thickness $s(t)$ has melted at any time t, while the prescribed heat input $Q(t)$ is applied at $x = s(t)$. The mathematical formulation of the problem is as follows [7, p. 190ff]:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho c\frac{\partial T}{\partial t}; \qquad s(t) < x < L, \qquad 0 < t < t_L; \tag{7}$$

$$T(x, 0) = 0; \qquad 0 \leq x \leq L; \tag{8}$$

$$\frac{\partial T}{\partial x}(L, t) = 0; \qquad 0 \leq t \leq t_L; \tag{9}$$

$$-k\frac{\partial T}{\partial x}(0, t) = Q(t); \qquad 0 \leq t \leq t_m; \tag{10a}$$

$$-k_m\frac{\partial T}{\partial x}[s(t), t] = Q(t) - \rho_m l\frac{ds}{dt}(t_+); \qquad t_m < t \leq t_L; \tag{10b}$$

$$T[s(t), t] = T_m; \qquad t_m \leq t \leq t_L; \tag{11}$$

$$T(x, t) < T_m; \qquad s(t) < x < L, \qquad 0 \leq t \leq t_L; \tag{11a}$$

$$s(t) = 0; \qquad 0 \leq t \leq t_m, \tag{12}$$

where the times t_m and t_L are respectively defined by the equations

$$T(0, t_m) = T_m \quad \text{and} \quad s(t_L) = L. \tag{13}$$

The thermal diffusivity $\kappa = k/(\rho c)$, the conductivity k, the specific heat c, and the density ρ are assumed to be functions of the temperature and therefore vary with both x and t. The subscript m, affixed to any of these quantities, indicates that the value at the melting temperature T_m must be used. The latent heat of melting is denoted by l.

It is easily shown[4] that, in general,

$$\int_0^t Q(t)\,dt = \int_{s(t)}^L H(T)\,dx + [\rho_m l + H_m]s(t), \tag{14}$$

where the heat content $H(T)$ is defined as[5]

[4]Derivations for the case of constant properties are given in [2, 7, 8]; they are readily extended to the present case if the heat content $H(T)$ is introduced as defined by Eq. (14a), since then the right-hand side of Eq. (7) reduces simply to $(\partial H/\partial t)$.

[5]This definition differs from the usual one by the inclusion of the density under the integral sign; cf. [9].

$$H(T) = \int_0^T \rho(T')c(T')\, dT'. \tag{14a}$$

Note that $\rho c > 0$ and therefore H is a monotonically increasing function of $T(x, t)$. The symbol H_m denotes $H(T_m)$.

Theorem III: Uniqueness of Solution. It will now be proved that there exists at most one solution[6] to Eqs. (7) to (13), corresponding to a prescribed function $Q(t)$. To prove this, assume that two distinct solutions exist, and denote them by the subscripts 1 and 2; then Theorem II insures uniqueness if $s_1 \equiv s_2$. Only the possibility $s_1 \not\equiv s_2$ need therefore be considered, or, without loss of generality, we may set

$$\begin{aligned} s_1 &= s_2 , & 0 \le t \le t', & \\ s_2 &> s_1 , & t' < t \le t' + \delta, & \quad \delta > 0. \end{aligned} \tag{15}$$

Write Eq. (14) for each solution, at some time t'' in the interval where $s_2 > s_1$, and subtract the results to get

$$0 = \int_{s_2(t'')}^{L} (H_2 - H_1)\, dx - \int_{s_1(t'')}^{s_2(t'')} H_1\, dx + (\rho_m l + H_m)[s_2(t'') - s_1(t'')]. \tag{15a}$$

However, since $T_2 = T_m$ and $T_1 \le T_m$ on $x = s_2(t)$, $t' < t \le t' + \delta$, it follows from Picone's Theorem [applied to the domain bounded by $t = t'$, $t = t' + \delta$, $x = s_2(t)$ and the reflection $x = 2L - s_2(t)$ of the latter line about $x = L$; cf. Fig. 1] that within this range of time $T_2 \ge T_1$ everywhere. Hence $H_2 \ge H_1$; since furthermore

$$\int_{s_1(t'')}^{s_2(t'')} H_1\, dx \le H_m[s_2(t'') - s_1(t'')], \tag{15b}$$

the right-hand side of Eq. (15a) cannot be zero. Hence conditions (15) cannot be met, and uniqueness is assured.

4. Upper and lower bounds. *Theorem IV.* Consider two solutions of Eqs. (7) to (13), respectively denoted by the subscripts 1 and 2, corresponding to two heat input functions $Q_1(t)$ and $Q_2(t)$ such that

$$Q_2(t) \ge Q_1(t). \tag{16}$$

Then

$$s_2(t) \ge s_1(t); \qquad 0 \le t \le t_{L2} \le t_{L1} , \tag{17a}$$

$$T_2(x, t) \ge T_1(x, t); \qquad s_2(t) \le x \le L. \tag{17b}$$

Proof. Eq. (14), written for $t = t_L$, gives

$$\int_0^{t_L} Q(t)\, dt = (\rho_m l + H_m)L. \tag{18}$$

Hence,

$$t_{2L} \le t_{1L} \tag{18a}$$

[6]That is, a twice continuously differentiable function $T(x,t)$ and a Lipschitz continuous function $s(t)$. Except in the special case of Eqs. (22), this character of the solution is assumed throughout the remainder of this paper.

or, in other words,

$$L = s_2(t_{L2}) \geq s_1(t_{L2}). \tag{18b}$$

No information concerning the relative magnitude of s_2 and s_1 at any other time can be immediately obtained from Eq. (14). The proof then proceeds in two steps, as follows: (a) it will first be shown that relation (17a) holds immediately, when s_1 and s_2 first differ, and then (b) that relation (17a) must then hold at all times. Inequality (17b) then follows as a corollary.

The proof given below holds also when the latent heat $l = 0$; in this limiting case however the proof of the present theorem follows readily from Theorem II.

The case of $Q_2 \equiv Q_1$ may here be omitted as trivial (Theorem III).

(a) Without loss of generality, assume that a time t' exists such that

$$\begin{aligned} Q_1(t) &= Q_2(t) \quad \text{for} \quad 0 \leq t \leq t', \\ Q_1(t) &< Q_2(t) \quad \text{for} \quad t' < t \leq t' + \delta, \qquad \delta > 0. \end{aligned} \tag{19}$$

It will now be shown that a number $\delta' > 0$ exists such that

$$s_2(t) > s_1(t) \quad \text{for} \quad t' < t \leq t' + \delta'. \tag{19a}$$

Assume in fact that this is not so; then, in view of (18b), there must be at least one time at which $s_2 = s_1$. Let the first of these times be t'' ($> t'$), as shown in Fig. 2.[7]

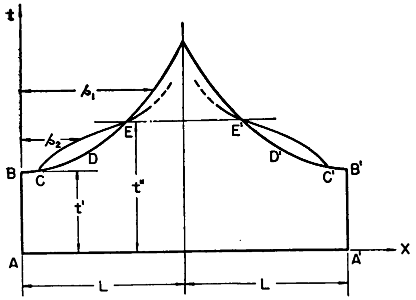

FIG. 2.

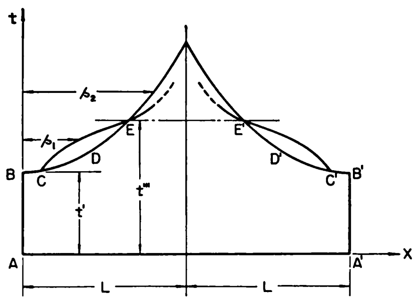

FIG. 3.

The quantity $(T_2 - T_1)$ is now zero along $CBAA'B'C'$, and is negative along CDE and $C'D'E'$; hence, by Picone's Theorem, it is non-positive along EE', and so therefore is the quantity $[H(T_2) - H(T_1)]$. But Eq. (14), written for both solutions gives, after subtraction,

$$0 \leq \int_0^{t''} (Q_2 - Q_1)\, dx = \int_{s_2 = s_1}^{L} [H(T_2) - H(T_1)]\, dx, \tag{19b}$$

which is evidently a contradiction; hence (19a) is proved.

(b) If now there exists some later time at which s_1 exceeds s_2, again there must be at least one instant at which $s_2 = s_1$; let the first of these times be t''' ($> t' + \delta'$), as shown in Fig. 3. The quantity $(T_2 - T_1)$ is zero along $CBAA'B'C'$, positive along CDE and $C'D'E'$, and zero at E and E'. By Picone's Theorem, it is thus non-negative along EE'. Eq. (10b) however gives, at point E of Fig. 3,

[7]The extended domain, including the reflection about $x = L$, is considered throughout the proof.

$$-k_m \frac{\partial(T_2 - T_1)}{\partial x} = (Q_2 - Q_1) - \rho_m l(s_2^{\cdot} - s_1^{\cdot}), \tag{20}$$

where dots indicate differentiation with respect to time. Clearly

$$s_2^{\cdot}(t''') \leq s_1^{\cdot}(t'''). \tag{21}$$

The right-hand side of (20) is positive if at least one of the inequalities (16) and (21) hold; if this is the case Eq. (20) implies $[\partial(T_2 - T_1)/\partial x] < 0$ at E, which (with $T_2 - T_1 = 0$ at E) would make $(T_2 - T_1)$ somewhere negative on EE' and thus leads to a contradiction.

In the special case

$$Q_1 = Q_2; \qquad s_1 = s_2; \qquad s_1^{\cdot} = s_2^{\cdot} \quad \text{at} \quad t = t''' \tag{22a}$$

the above proof fails since then $[\partial(T_2 - T_1)/\partial x] = 0$ at E. We then assume

$$\left.\begin{aligned} s(t) &= \frac{(t - t''')^2}{2!} s^{\cdot\cdot}(t'') + \frac{(t - t''')^3}{3!} s^{\cdot\cdot\cdot}(t''') + \cdots \\ Q(t) &= (t - t''')Q^{\cdot}(t''') + \frac{(t - t''')^2}{2!} Q^{\cdot\cdot}\, t''') + \cdots \end{aligned}\right\} t \geq t''', \tag{22b}$$

where clearly

$$\begin{aligned} Q_2^{\cdot}(t''') &\geq Q_1^{\cdot}(t''') \\ s_2^{\cdot\cdot}(t''') &\leq s_1^{\cdot\cdot}(t''') \end{aligned} \tag{22c}$$

Differentiation of (10b) along s gives[8]

$$\begin{aligned} -k_m\left(\frac{\partial^2 T}{\partial x^2} s^{\cdot} + \frac{\partial^2 T}{\partial x \partial t}\right) - \left(\frac{dk}{dT}\right)_m \left(\frac{\partial T}{\partial x} s^{\cdot} + \frac{\partial T}{\partial t}\right) \frac{\partial T}{\partial x} &= \frac{dQ}{dt} - \rho_m l \frac{d^2 s}{dt^2} \\ &\quad - l\left(\frac{d\rho}{dT}\right)_m \left(\frac{\partial T}{\partial x} s^{\cdot} + \frac{\partial T}{\partial t}\right) \frac{ds}{dt}, \end{aligned} \tag{22d}$$

while Eq. (11), written in differential form, is

$$\frac{\partial T}{\partial x} s^{\cdot} + \frac{\partial T}{\partial t} = 0 \tag{22e}$$

or, with (7),

$$\frac{\partial T}{\partial x} s^{\cdot} + \frac{1}{\rho c}\left[\frac{dk}{dT}\left(\frac{\partial T}{\partial x}\right)^2 + k \frac{\partial^2 T}{\partial x^2}\right] = 0 \tag{22f}$$

Writing Eq. (22f) for solutions 1 and 2 and subtracting one obtains for this case $[\partial^2(T_2 - T_1)/\partial x^2] = 0$ at E. The same process applied to (22d) now gives[9], at E,

$$-k_m \kappa_m \frac{\partial^3(T_2 - T_1)}{\partial x^3} = (Q_2^{\cdot} - Q_1^{\cdot}) - \rho_m l(s_2^{\cdot\cdot} - s_1^{\cdot\cdot}) \tag{22g}$$

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[8]In the equations which follow, all quantities must be evaluated at $x = s(t)$.

[9]After use of Eq. (7) to give

$$\frac{\partial^2 T}{\partial x \partial t} = \frac{\partial}{\partial x}\left[\frac{1}{\rho c}\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right] = \kappa \frac{\partial^3 T}{\partial x^3} + f\left(\frac{\partial^2 T}{\partial x^2}, \frac{\partial T}{\partial x}\right).$$

Unless both equality signs hold in (22c), the right-hand side of (22g) is positive; but this implies $[\partial^3(T_2 - T_1)/\partial x^3] < 0$ at E, which again leads to a contradiction since it makes $(T_2 - T_1)$ somewhere negative on EE'. The case

$$\left.\begin{array}{lll} Q_1 = Q_2 , & \dot{Q}_1 = \dot{Q}_2 , & \\ s_1 = s_2 , & \dot{s}_1 = \dot{s}_2 , & \ddot{s}_1 = \ddot{s}_2 \end{array}\right\} \text{ at } t = t''' \tag{23}$$

must still be investigated, as well as subsequent special cases of this type. These are treated by means of further differentiation of Eqs. (22d), (22e) and (7), along s; but inspection reveals that, in general, the result will be (at E):

$$-k_m\kappa_m \frac{\partial^{2n+1}(T_2 - T_1)}{\partial x^{2n+1}} = \frac{d^n(Q_2 - Q_1)}{dt^n} - \rho_m l \frac{d^{n+1}(s_2 - s_1)}{dt^{n+1}} \tag{24a}$$

$$\frac{\partial^m(T_2 - T_1)}{\partial x^m} = 0; \qquad m = 0, 1, 2, \cdots, 2n \tag{24b}$$

when

$$\frac{d^m(Q_2 - Q_1)}{dt^m} = 0; \qquad m = 0, 1, 2, \cdots, n - 1, \tag{24c}$$

$$\frac{d^m(s_2 - s_1)}{dt^m} = 0; \qquad m = 0, 1, 2, \cdots, n. \tag{24d}$$

Choose n as the smallest integer for which at least one term of the right-hand side of (24a) does not vanish, and then note that, just as with Eqs. (20) and (22g), this contradicts the non-negative character of $(T_2 - T_1)$ required by Picone's Theorem along EE'. The proof of the theorem is thus complete.

It may be noted that the converse of this theorem is false, that is, the validity of Eqs. (17) does not imply the validity of (16).

5. Procedure for the construction of bounds. The theorem developed in the preceding section is particularly useful in the solution of melting problems of the type being considered here, because, as may be readily observed, *any* solution of Eq. (7) valid within the original slab thickness $0 < x < L$ is a solution of the melting problem for a particular set of functions $Q(t)$ and $s(t)$. Indeed, such a solution of (7) satisfies all the equations of the formulation of Section 3 with the exception of Eqs. (10b) and (11): the latter may be used to determine $s(t)$ and the former to calculate $Q(t)$. This observation forms in fact the basis for the integral-equations method devised in [1] for the solution of this melting problem.

Solutions of the heat-conduction equation (7), thermal properties being uniform, are easily constructed in terms of the fundamental solution:

$$\begin{aligned} T_0(x, t) &= \frac{2\sqrt{\kappa t}}{k} \sum_{n=0}^{\infty} \left[\text{ierfc}\left(\frac{2nL + x}{2\sqrt{\kappa t}}\right) + \text{ierfc}\left(\frac{2(n+1)L - x}{2\sqrt{\kappa t}}\right)\right] \\ &= \frac{L}{k}\left\{\frac{\kappa t}{L^2} + \frac{3x^2 - 6Lx + 2L^2}{L^2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\exp[-\kappa n^2\pi^2 t/L^2]}{n^2} \cos\frac{n\pi x}{L}\right\} \end{aligned} \tag{25}$$

which satisfies the conditions

$$-k\frac{\partial T_0}{\partial x}(0, t) = 1, \qquad \frac{\partial T_0}{\partial x}(L, t) = 0, \qquad T_0(x, 0) = 0. \tag{25a}$$

Duhamel's theorem may now be used to find the solution for any prescribed heat input $Q^*(t)$ at $x = 0$. Since the actual heat input there is known up to the onset of melting, it may be often convenient to choose $Q^*(t) = Q(t)$ for $t \leq t_m$, and arbitrary thereafter. Hence the temperature $T^*(x, t)$ due to Q^* is [1, 7, 9]:

$$T^*(x, t) = \begin{cases} \int_0^t Q(t_1) \frac{\partial T_0}{\partial t_1}(x, t - t_1)\, dt_1 = T(x, t), \quad t \leq t_m, \\ \int_0^{t_m} Q(t_1) \frac{\partial T_0}{\partial t_1}(x, t - t_1)\, dt_1 + \int_{t_m}^t Q^*(t_1) \frac{\partial T_0(x, t - t_1)}{\partial t_1}\, dt_1, \ t \geq t_m. \end{cases} \tag{26}$$

The procedure then consists of calculating T^* for suitable choices of Q^*, and evaluating $Q(t)$ and $s(t)$ for each after melting has begun; Theorem IV then insures that the melting rate s so calculated is lower than the actual one in the range $t_m \leq t \leq t^*$ if $Q^* \leq Q$ in that range, and higher if $Q^* \geq Q$ in that range. An example of this procedure may be found in the next section of this paper.

It may be remarked here that (as obviously follows from Theorem II applied to the region $0 < x < L$), if two fictitious heat inputs $Q_1^*(t)$ and $Q_2^*(t)$ are considered, such that (on $x = 0$)

$$Q_2^*(t) \geq Q_1^*(t), \tag{27}$$

then the thickness melted $s(t)$ corresponding to Q_2^* is larger than that corresponding to Q_1^*. This fact gives however no clue as to the relation between the corresponding heat inputs $Q_2(t)$ and $Q_1(t)$ at $x = s(t)$; to obtain this Theorem IV is required.

6. Numerical example. As a specific example, consider the half-space $x > 0$ which melts under a constant uniform heat input Q_0; this is the problem for which the exact solution was calculated in [8] by means of a high-speed computer. Before melting the solution is

$$\frac{T(x, t)}{Q_0} = T_0(x, t) = \frac{2(\kappa t)^{1/2}}{k} \operatorname{ierfc} \frac{x}{2(\kappa t)^{1/2}} \tag{28}$$

and

$$t_m = \left(\frac{kT_m}{2Q_0}\right)^2 \frac{\pi}{\kappa} \tag{28a}$$

Choose for the fictitious heat input Q^* a piecewise constant function as follows:

$$Q^*(t) = \alpha_i Q_0 t_i < t < t_{i+1}, \qquad i = 0, 1, 2, \cdots \tag{29}$$

with

$$t_0 = 0, \qquad t_1 = t_m, \qquad t_{i+1} > t_i;\ \alpha_0 = 1. \tag{29a}$$

Then Eq. (26) gives the temperature T^* during the period $t_n < t < t_{n+1}$ as

$$T^*(x, t) = Q_0 \sum_{i=0}^{n} (\alpha_i - \alpha_{i-1}) T_0(x, t - t_i); \qquad a_{-1} = 0. \tag{30}$$

This temperature satisfies all Eqs. (7) to (12), with the exception of (10b) and (11). With the notation [1]:

$$y = \frac{t}{t_m} - 1; \qquad \xi(y) = \frac{Q_0\, s}{\pi^{1/2} k T_m}; \qquad y_i = \frac{t_i}{t_m} - 1. \tag{31}$$

Eqs. (11) and (10b) reduce respectively (for $y_n < y < y_{n+1}$) to

$$1 = \sum_{i=0}^{n} (\alpha_i - \alpha_{i-1})[\pi(y - y_i)]^{1/2} \operatorname{ierfc} \frac{\xi(y)}{(y - y_i)^{1/2}} \tag{32a}$$

and (for $l = 0$) to

$$\frac{Q(y)}{Q_0} = \sum_{i=0}^{n} (\alpha_i - \alpha_{i-1}) \operatorname{erfc} \frac{\xi(y)}{(y - y_i)^{1/2}} \tag{32b}$$

respectively. The first of these is easily solved numerically for $\xi(y)$, for any choice of the α_i's and the y_i's; with each value of $\xi(y)$, $Q(y)$ is then readily obtained from (32b). The choice of the α_i's and of the y_i's is made, as the calculations progress, so as to keep $Q(y)/Q_0$ either larger or smaller than unity, depending on whether an upper or a lower bound is desired. Numerical results for both bounds are shown in Fig. 4: the upper bound is calculated with $n = 3$ and with

$$\begin{aligned} \alpha_0 &= 1, & y_0 &= -1, \\ \alpha_1 &= 1.2, & y_1 &= 0, \\ \alpha_2 &= 1.4, & y_2 &= .37, \\ \alpha_3 &= 1.6, & y_3 &= .6, \end{aligned} \tag{33a}$$

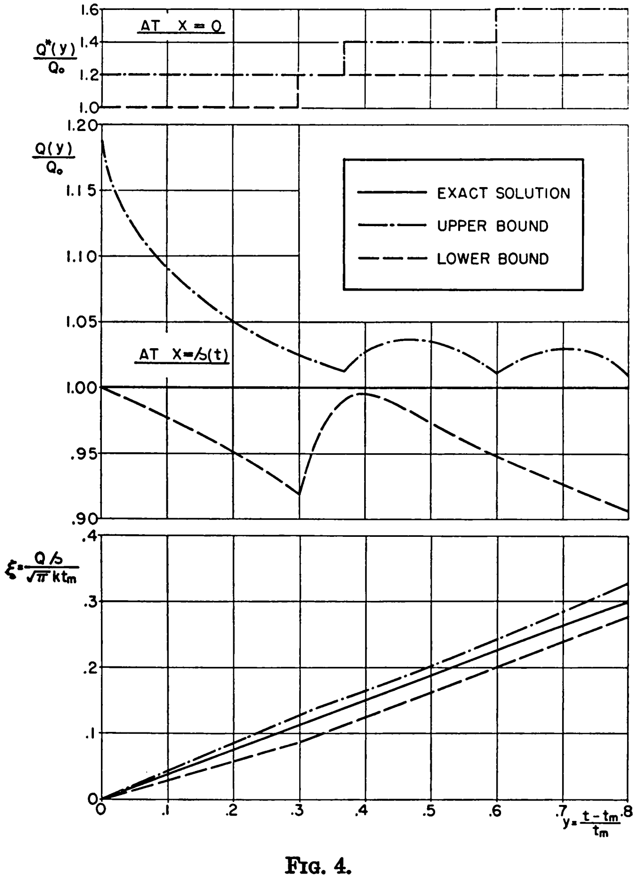

FIG. 4.

while for the lower bound $n = 1$ and

$$\alpha_0 = 1, \qquad y_0 = -1, \qquad\qquad (33b)$$
$$\alpha_1 = 1.2, \qquad y_1 = .3.$$

The above choices may be seen to give reasonable accuracy for $y < 0.8$; a different set of choices to give greater accuracy could of course be devised without any difficulty.

The curves for $Q(y)$ in Fig. 4 appear to have discontinuous derivatives at all values of $y_i > 0$; in reality all derivatives are continuous there, but the curves exhibit sharp turns in transition zones too short to be evident in the graph with the scale used. It can also be shown from Eq. (32a) that, for the upper bound calculated with $n = 1$ and $y_1 = 0$,

$$\xi(y) = ay^{1/2} \quad \text{for} \quad 0 < y \ll 1 \qquad (34)$$

where a is the root of the transcendental equation

$$a = (\alpha_1 - 1) \text{ ierfc } a \qquad (34a)$$

From Eq. (32b) it then follows that

$$\frac{Q(0)}{Q_0} = 1 + (\alpha_1 - 1) \text{ erfc } a \qquad (34b)$$

The exact solution obtained in [8] for the position $\xi(y)$ of the moving boundary is included in the lower graph of Fig. 4 for purposes of comparison.

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