AN UNSTEADY PROBLEM IN MAGNETOHYDRODYNAMICS[1]

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Consider an infinite mass of incompressible, electrically conducting fluid which is penetrated by a uniform magnetic field H **i** and subjected to disturbances of order ϵ, where

$$\epsilon \ll 1 \tag{1}$$

Writing

$$\frac{\mathbf{q}'}{\omega L} = \epsilon \mathbf{q}(x_i, t) + o(\epsilon),$$

$$\frac{\mathbf{h}'}{RmH} = \epsilon \mathbf{h}(x_i, t) + o(\epsilon),$$

$$\frac{p' - p'_\infty}{\rho\omega^2 L^2} = \epsilon p(x_i, t) + o(\epsilon),$$

where $\mathbf{q}'$ is the velocity, $\mathbf{h}'$ the magnetic field perturbation, and p' the pressure, and inserting these expansions into the magnetohydrodynamic equations (Ref. 1), we have, collecting terms of order ϵ,

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla p + N\mathbf{i} \times \nabla \times \mathbf{h} = \frac{1}{Re} \nabla^2 \mathbf{q}, \tag{2}$$

$$\mathbf{i} \cdot \nabla \mathbf{q} + \nabla^2 \mathbf{h} = Rm \frac{\partial \mathbf{h}}{\partial t}, \tag{3}$$

$$\nabla \cdot \mathbf{h} = \nabla \cdot \mathbf{q} = 0, \tag{4}$$

where $x_i = x_i'/L$, $t = \omega t'$. The dimensionless parameters appearing in the above relations are defined in the usual notation by

$$Re = \frac{\omega L^2}{\nu}, \qquad Rm = \sigma\mu\omega L^2, \qquad N = \frac{\mu^2 \sigma H^2}{\rho\omega},$$

where σ is the electrical conductivity, μ the magnetic permeability, ν the kinematic viscosity, and ρ the density, expressed in *mksq* units. We interpret ω and L as some characteristic frequency and length associated with the disturbances.

With the additional conditions

$$Re \gg 1, \qquad Rm \ll 1, \tag{5}$$

the terms on the right side of (2) and (3) become nominally small and will be neglected here. For strictly two-dimensional flow, $x_1 = x$, $x_2 = y$, and $\mathbf{q} = u\mathbf{i} + v\mathbf{j}$, we then obtain the system

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$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \tag{6}$$

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} + Nv = 0, \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8}$$

for the velocity and pressure.

Suppose now that disturbances are set up in the fluid by a vertical flat plate of width $2L$ executing a simple harmonic oscillation through a distance $2\,\epsilon\, L$ parallel to the x-axis, the frequency of the oscillation being ω. For this problem we look for a solution of (6), (7), and (8) satisfying

$$u = e^{it} \quad \text{on} \quad x = 0, \qquad -1 < y < +1, \tag{9}$$

$$u = v = p = 0 \qquad \text{at infinity}\ . \tag{10}$$

Note that the exact boundary condition on $x = 0$ is here replaced by its representation for small ϵ.

A solution of this boundary value problem is

$$p(x, y, t) = P(x, y, t; \lambda) = i\lambda \exp{(it)}\phi(x/\lambda, y),$$

$$u = \frac{\partial A}{\partial y}, \qquad v = -\frac{\partial A}{\partial x}; \qquad A(x, y, t; \lambda) = \exp{(it)}\psi(x/\lambda, y),$$

where

$$\lambda = (1 - iN)^{1/2}$$

and

$$\phi(x, y) - i\psi(x, y) = f(z) = (z^2 + 1)^{1/2} - z; \qquad z = x + iy$$

is the complex (perturbation) potential for steady, irrotational flow past a vertical plate. The uniqueness of the solution is apparent by comparison, for $N = 0$, with that classical problem.

The magnetic field satisfies

$$\nabla^2 \mathbf{h} = -\frac{\partial \mathbf{q}}{\partial x}, \tag{11}$$

$$\mathbf{h} \quad \text{continuous on} \quad x = 0, \qquad -1 < y < +1, \tag{12}$$

$$\mathbf{h} = 0 \quad \text{at infinity}, \tag{13}$$

where we have taken the boundary to be insulated. The relevant solution of (11) will consist of two parts. We set

$$\mathbf{h} = \mathbf{h}^{(1)} + \mathbf{h}^{(2)}, \qquad \mathbf{h}^{(1)} = h_1^{(1)}\mathbf{i} + h_2^{(1)}\mathbf{j}, \qquad \text{etc.}$$

where $\mathbf{h}^{(1)}$ is a particular integral of (11):

$$h_1^{(1)} = -\frac{1}{N} P(x, y, t; \lambda),$$

$$h_2^{(1)} = -\frac{i}{N} A(x, y, t; \lambda).$$

The harmonic component $\mathbf{h}^{(2)}$ is uniquely determined by (12) and (13) and renders $\mathbf{h}$ regular at all points of the (x, y) plane, excluding the branch points of $f(z)$. Thus

$$h_1^{(2)} = \frac{i\lambda}{N} \exp (it)\phi(x, y),$$

$$h_2^{(2)} = \frac{i\lambda}{N} \exp (it)\psi(x, y).$$

These results simplify considerably if N is large compared to 1. In this case x and N appear in the pressure and stream function only in the combination $x/N^{1/2}$ to the first approximation. Consequently, given any positive constant C and a number $\gamma < \frac{1}{2}$, condition (9) may be approximated as closely as is desired at all points in the rectangle $-CN^{\gamma} < x < +CN^{\gamma}$, $-1 < y < +1$, and hence, also on any convex insulated boundary situated within this rectangle, as $N \rightarrow \infty$. Furthermore, if ϵ is *large*, but $\epsilon/N^{1/2}$ is small, the approximation to velocity and pressure remains valid in the sense that our solutions still satisfy the exact equations and conditions to order unity. Insertion of the approximate solution into the full equations provides an estimate of the error (and of the relative magnitude of the nonlinear terms) as $O(\epsilon/N^{1/2})$ at all points of regularity.

No similar conclusions may be drawn concerning $\vec{\mathbf{h}}$ since, regardless of how large we make N, the harmonic component will involve the solution of an interior problem formulated in the (x, y) plane for the boundary in question.

Among the properties of these solutions which may be of interest, we observe that, for sufficiently large N, the amplitude of the oscillation in the velocity becomes arbitrarily large at any finite point $(x, \pm 1)$. On the other hand, for fixed N and at sufficiently distant points, oscillations in the magnetic field have the greater amplitude. The total force per unit width experienced by the body, $F(t)$, may be determined by the boundary values of the pressure and is given by

$$F(t) = \epsilon\omega^2 LM_0(N^2 + 1)^{1/4} \exp [i(t + \theta/2)] \tag{14}$$

to the first approximation. Here

$$\theta = \pi/2 + \tan^{-1}(1/N),$$

and

$$M_0 = \rho\pi L^2$$

is the apparent mass of the plate in the absence of a field. Thus, the gross effect of a strong field is to increase the apparent mass roughly as $N^{1/2}$ and to shift the phase by approximately $\pi/4$.

With minor changes our procedure may be applied to analogous problems with axially symmetric boundary conditions. Omitting these results, we shall note only that (14) provides the correct expression for the force experienced by an oscillating circular disk of radius L, with M_0 taken as the apparent mass for the disk in the absence of a field.

Reference

1. Cowling, T. G., *Magnetrohydrodynamics*, Interscience Publishers, Inc., New York, 1957