A NOTE ON INPUT-OUTPUT SPECTRAL DENSITIES*

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If stationary noise $x(t)$ is applied to a linear time invariant system $\mathcal{S}$ with transfer function $H(j\omega)$, then the mean-square value σ^2 of the output noise $z(t)$ is given by the well known formula

$$\sigma^2 = \int_0^\infty |H(j\omega)|^2 w_x(f)\, df \tag{1}$$

where $w_x(f)$ is the spectral density of the input process $x(t)$. Apparently (1) has no simple analog when $x(t)$ is non-stationary. Of course, one can always write, even in the non-stationary case,

$$\sigma^2(t) = \mathcal{E}z^2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t - \xi)h(t - \zeta)\psi_x(\xi, \zeta)\, d\xi\, d\zeta$$

where $h(t)$ is the impulsive response of $\mathcal{S}$ and $\psi_x(\xi, \zeta) = \mathcal{E}x(\xi)x(\zeta)$ is the correlation function of $x(t)$. However, in certain practical problems the formulation embodied in (1) is more convenient.

In this note we wish to deduce a formula similar to (1) in the case where the input $y(t)$ to $\mathcal{S}$ is of the form

$$y(t) = g(t)x(t) \tag{2}$$

and $x(t)$ is stationary noise while the modulation function $g(t)$ is of the form

$$g(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n). \tag{3}$$

Such a situation may arise, for example, when a sinusoidal signal, corrupted by additive stationary noise, is applied to a square law detector followed by a linear filter $\mathcal{S}$. If $K \cos \Omega t + \eta(t)$, where $\eta(t)$ is the noise, is applied to the square law detector, one of the terms in its output will be $(2K \cos \Omega t)\eta(t)$—which is of the form (2). Thus, for example, if one wished to compute the signal-to-noise ratio at the output of the linear filter $\mathcal{S}$, it would be necessary to compute (among other terms) the variance of the output of $\mathcal{S}$ corresponding to the input $y(t)$. This is precisely the problem we have set ourselves.

Returning to (2) we see from the definition of $g(t)$, cf. (3), that $y(t)$ is non-stationary, but of special form. We shall show that if $z(t)$ is the output of $\mathcal{S}$ corresponding to this input $y(t)$, then the mean-square value of the output noise $z(t)$ is

$$\sigma^2(t) = \frac{1}{4} \int_0^\infty |M(\omega) + \{M(-\omega)\}_C|^2 w_x(f)\, df, \tag{4}$$

where the subscript C indicates the conjugate complex quantity,

$$M(\omega) = \sum_{n=1}^{N} a_n H(j(\omega + \omega_n))e^{j(\omega_n t + \phi_n)}, \tag{5}$$

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and $H(j\omega)$ is the transfer function of $\mathcal{S}$. Certain generalizations can also be obtained.

We begin by writing the output noise $z(t)$ of $\mathcal{S}$ arising from the input $y(t)$ as

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)Y(j\omega)e^{i\omega t}\,d\omega$$

where $Y(j\omega)$ is the Fourier transform of $y(t)$:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}\,dt. \tag{6}$$

A difficulty arises since $Y(j\omega)$ will not exist in the classical sense unless $g(t)e^{-i\omega t}$ is absolutely integrable—which it is not. This problem, as well as the subsequent use of the Dirac delta functional could be avoided. For example, we could replace a_n in (3) by $a_n e^{-\alpha|t|}$ with $\alpha > 0$, and then investigate the value of σ^2 as α approaches zero; or we could let $a_n = 0$ for $|t| > T$ and investigate the value of σ^2 as T increases without limit. However, it is more convenient to use the approach we employ below.

The mean-square value of the output noise is given by

$$\sigma^2(t) = \mathcal{E}z^2(t) = \mathcal{E}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)Y(j\omega)Y(j\beta)e^{i\omega t}e^{i\beta t}\,d\omega\,d\beta. \tag{7}$$

Substituting the right-hand side of (6) for $Y(j\omega)$ and $Y(j\beta)$ in (7) we obtain

$$\sigma^2(t) = \frac{1}{4\pi^2}\mathcal{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)y(\xi)y(\zeta)e^{-i\omega\xi}e^{-i\beta\zeta}e^{i\omega t}e^{i\beta t}\,d\omega\,d\beta\,d\xi\,d\zeta. \tag{8}$$

Now,

$$\mathcal{E}y(\xi)y(\zeta) = g(\xi)g(\zeta)\psi_x(\xi - \zeta),$$

where ψ_x is the correlation function of $x(t)$. Thus (8) may be written as

$$\sigma^2(t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)e^{i\omega t}e^{i\beta t}X(\omega, \beta)\,d\omega\,d\beta, \tag{9}$$

where

$$X(\omega, \beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi)g(\zeta)\psi_x(\xi - \zeta)e^{-i\omega\xi}e^{-i\beta\zeta}\,d\xi\,d\zeta.$$

Since

$$g(\xi)g(\zeta) = \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m \cos(\omega_n\xi + \phi_n)\cos(\omega_m\zeta + \phi_m),$$

we may write

$$X(\omega, \beta) = \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m X_{nm}(\omega, \beta), \tag{10}$$

where

$$X_{nm}(\omega, \beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(\omega_n\xi + \phi_n)\cos(\omega_m\zeta + \phi_m)\psi_x(\xi - \zeta)e^{-i\omega\xi}e^{-i\beta\zeta}\,d\xi\,d\zeta.$$

If we make the change of variable $u = \xi - \zeta$, $v = \zeta$ with unity Jacobian in $X_{nm}(\omega, \beta)$, we obtain

$$X_{nm}(\omega, \beta) = \int_{-\infty}^{\infty} \cos(\omega_m v + \phi_m)e^{-iv(\omega+\beta)}\,dv\int_{-\infty}^{\infty} \cos(\omega_n u + \omega_n v + \phi_n)\psi_x(u)e^{-i\omega u}\,du. \tag{11}$$

The second integral is readily evaluated as

$$\int_{-\infty}^{\infty} \cos(\omega_n u + \omega_n v + \phi_n)\psi_x(u)e^{-i\omega u}\,du$$
$$= \tfrac{1}{4}w_x(f + f_n)e^{-i(\omega_n v+\phi_n)} + \tfrac{1}{4}w_x(f - f_n)e^{i(\omega_n v+\phi_n)}, \tag{12}$$

where $f = \omega/2\pi$, $f_n = \omega_n/2\pi$, $1 \le n \le N$, and

$$w_x(f) = 2\int_{-\infty}^{\infty} \psi_x(u)e^{-i\omega u}\,du$$

is the spectral density of $x(t)$. Substituting (12) in (11) we obtain

$$X_{nm}(\omega, \beta) = \frac{\pi}{4} w_x(f + f_n)[\delta(\omega + \beta + \omega_n - \omega_m)e^{i(\phi_m-\phi_n)} + \delta(\omega + \beta + \omega_n + \omega_m)e^{-i(\phi_m+\phi_n)}]$$
$$+ \frac{\pi}{4} w_x(f - f_n)[\delta(\omega + \beta - \omega_n - \omega_m)e^{i(\phi_m+\phi_n)} + \delta(\omega + \beta - \omega_n + \omega_m)e^{-i(\phi_m-\phi_n)}]. \tag{13}$$

From (10), we may write σ^2 [cf. (9)] as

$$\begin{aligned}\sigma^2 &= \frac{1}{4\pi^2}\sum_{n,m} a_n a_m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)e^{i\omega t}e^{i\beta t}X_{nm}(\omega, \beta)\,d\omega\,d\beta \\ &= \frac{1}{4\pi^2}\sum_{n,m} a_n a_m Y_{nm},\end{aligned} \tag{14}$$

where

$$Y_{nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)e^{i\omega t}e^{i\beta t}X_{nm}(\omega, \beta)\,d\omega\,d\beta.$$

Equation (13) may now be used to write Y_{nm} as

$$\begin{aligned}Y_{nm} = \frac{\pi}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & H(j\omega)H(j\beta)e^{i\omega t}e^{i\beta t}w_x(f + f_n)[\delta(\omega + \beta + \omega_n - \omega_m)e^{i(\phi_m-\phi_n)} \\ & + \delta(\omega + \beta + \omega_n + \omega_m)e^{-i(\phi_m+\phi_n)}]\,d\omega\,d\beta, \\ & + \frac{\pi}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(j\omega)H(j\beta)e^{i\omega t}e^{i\beta t}w_x(f - f_n)[\delta(\omega + \beta - \omega_n - \omega_m)e^{i(\phi_m+\phi_n)} \\ & + \delta(\omega + \beta - \omega_n + \omega_m)e^{-i(\phi_m-\phi_n)}]\,d\omega\,d\beta.\end{aligned}$$

Replacing $\omega + \omega_n$ by ω in the first integral and $\omega - \omega_n$ by ω in the second integral enables us to write Y_{nm} as

$$\begin{aligned}Y_{nm} = \frac{\pi}{4}\int_{-\infty}^{\infty} & H(j(\omega - \omega_n))H(-j(\omega - \omega_m))e^{i(\omega_m-\omega_n)t}w_x(f)\,d\omega\,e^{i(\phi_m-\phi_n)} \\ & + \frac{\pi}{4}\int_{-\infty}^{\infty} H(j(\omega - \omega_n))H(-j(\omega + \omega_m))e^{-i(\omega_m+\omega_n)t}w_x(f)\,d\omega\,e^{-i(\phi_n+\phi_m)} \\ & + \frac{\pi}{4}\int_{-\infty}^{\infty} H(j(\omega + \omega_n))H(-j(\omega - \omega_m))e^{i(\omega_n+\omega_m)t}w_x(f)\,d\omega\,e^{i(\phi_m+\phi_n)} \\ & + \frac{\pi}{4}\int_{-\infty}^{\infty} H(j(\omega + \omega_n))H(-j(\omega + \omega_m))e^{i(\omega_n-\omega_m)t}w_x(f)\,d\omega\,e^{-i(\phi_m-\phi_n)}.\end{aligned}$$

This result and (14) then imply

$$\sigma^2 = \frac{1}{4}\int_0^\infty [M(\omega)M(-\omega) + M(\omega)\{M(\omega)\}_C + \{M(-\omega)\}_C M(-\omega) + \{M(-\omega)M(\omega)\}_C] w_x(f)\, df, \qquad (15)$$

which immediately reduces to (4).

If in place of equation (7) we start with

$$\psi_z(t;\tau) = \mathcal{E}z(t)z(t+\tau)$$

and employ the Wiener-Khintchine relations[1] to write

$$w_z(t;f) = 2\int_{-\infty}^\infty \psi_z(t;\tau)e^{-i\omega\tau}\, d\tau$$

as the time dependent spectral density, then using the above techniques we can show that

$$w_z(t;f) = \frac{1}{4}H(j\omega)\left\{\sum_{m=1}^N a_m[\{M(\omega-\omega_m)\}_C + M(-\omega+\omega_m)]e^{i(\omega_m t+\phi_m)}w_x(f-f_m) + \sum_{m=1}^N a_m[\{M(\omega+\omega_m)\}_C + M(-\omega-\omega_m)]e^{-i(\omega_m t+\phi_m)}w_x(f+f_m)\right\}. \qquad (16)$$

It is easily seen, then, that

$$\sigma^2 = \frac{1}{2}\int_{-\infty}^\infty w_z(t;f)\, df$$

where σ^2 is given by (4).

[1]D. G. Lampard, *Generalization of the Wiener-Khintchine theorem to nonstationary processes*, J. Appl. Phys. **25**, 802-803(1954)

Correction to the paper

DUALITY IN NONLINEAR PROGRAMMING

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There is an incorrect statement of a previous result. In particular the last sentence of the Converse Duality Theorem should read:

"If $\varphi(x)$ is quadratic and if $g(x)$ is linear, then a weaker converse theorem is also true if $\varphi(x)$ is merely convex and twice continuously differentiable."