PARTIALLY CONSTRAINED IMPINGING JETS*

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1. Introduction. Treatises on hydrodynamics generally contain discussions of two standard examples of plane incompressible jet flows; viz.:

(i) Efflux of liquid with high pressure at infinity from a reservoir with straight walls into a jet surrounded by stagnant fluid at lower pressure;

(ii) Impact of two jets, in which the moving fluid is surrounded by four regions of stagnant fluid, all at the same pressure.

Contemplation of these examples suggests the problem, to determine the following flow:

(iii) Impact of two jets, each partially constrained on one side by straight walls, in which the jets are also partially bounded by four regions of stagnant fluid, not all at the same pressure.

The symmetrical form of (iii), shown schematically in Fig. 1 (with an inflection point A_4 on the high pressure boundary), will be discussed in this note. A perfectly obvious generalization of (iii) is

(iv) Impact of two jets, each partially constrained on both sides by straight walls, in which the jets are also bounded by four regions of stagnant fluid, not all at the same pressure.

The symmetrical form of this flow is discussed at the end of this note by straightforward modifications of the mathematical apparatus used to describe (iii).

The flows to be constructed are interesting for their own sakes as relatively simply explicitly describable jets. Additional interest might be stimulated by the following considerations. Flows of type (ii) have been used to suggest a qualitative explanation of the motion of the liner of a shaped charge. In reality, the motion is produced by the difference between the high pressure in the detonation products on one side of the liner and atmospheric pressure on the other side. Furthermore, at an intermediate stage only part of the liner has collapsed or begun to collapse, while the rest is still rigid. In an admittedly imperfect way flows (iii) or (iv) more nearly approximate these features than (ii).

We note also that various pressures, velocities, and widths are referred to in Fig. 1, and four additional parameters will appear in the discussion in the following sections. Let us suppose we have experimental measurements of the angles θ_3, θ_4, the three widths h_1, h_3, and h_5, and assume we know p_1 (which might be atmospheric pressure) and the density ρ of the jets. Then we could apply the seven equations (2.3), (3.4)—(3.6), (4.4), (4.6), and (4.8) to determine the four relatively uninteresting mathematical parameters and the three important physical parameters p_5, U_1, and U_5.

2. Mathematical formulation of problem. Plane irrotational incompressible flow can be characterized by a complex potential function

$$\Phi(z) = \phi + i\psi. \tag{2.1}$$

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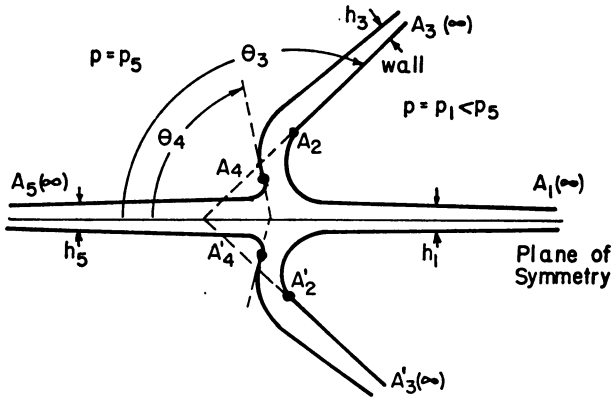

FIG. 1. Flow in physical (z) plane.

Here $\Phi(z)$ is an analytic function of the complex variable $z = x + iy$, ϕ is the velocity potential function, ψ the stream function, and

$$w = u - iv = d\Phi/dz \tag{2.2}$$

is the complex velocity. The pressure p within the jets is determined by Bernoulli's equation

$$p + \tfrac{1}{2}\rho \mid w \mid^2 = \text{constant}. \tag{2.3}$$

Conditions on the jet boundaries are characterized by

$$p = p_\alpha, \qquad \alpha = 1, 5, \tag{2.4}$$

and thus $\mid w \mid$ assumes corresponding constant values

$$\mid w \mid = U_\alpha, \qquad \alpha = 1, 5. \tag{2.5}$$

To seek $\Phi(z)$ or $w(z)$ directly in the z-plane is hopeless, since the location of the jet boundaries characterized by (2.4) or (2.5) is not known *a priori*. The classical artifice for overcoming this difficulty is to invert (2.2) to determine

$$z = f(w), \tag{2.6}$$

where f is an analytic function of w. Then straight-streamlines (walls, or the axis of symmetry) have as their images in the w—or hodograph-plane segments of lines through the origin, and free jet boundaries correspond to arcs of circles (2.5) with centers at the origin. Thus the impinging jets, of Fig. 1 map onto the interior of the region shown in Fig. 2. The circular cuts A_3A_4 and $A_3'A_4'$ appear as a matter of necessity, while the cut $A_5A_6A_5'$ has been introduced for later convenience.

Boundary conditions and other relevant properties of $f(w)$ can be determined as follows. On the straight streamlines of Fig. 1 dz is parallel to $\bar{w}$ and dw is parallel to w. Thus on all straight segments shown in Fig. 2

$$\text{Im } w^2\, dz/dw = \text{Im } w^2 f' = 0. \tag{2.7}$$

On the free jet boundaries, which are also streamlines dz is again parallel to $\bar{w}$, and on the circular arcs (2.5) dw is parallel to iw. Thus on all circular arcs shown in Fig. 2

$$\text{Re } w^2\, dw/dz = \text{Re } w^2 f'(w) = 0. \tag{2.8}$$

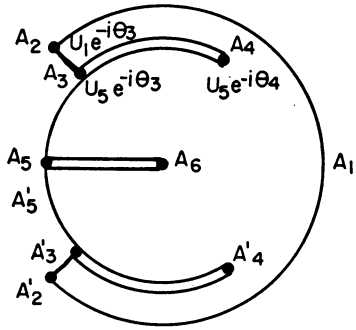

FIG. 2. Map in hodograph (w) plane.

To guarantee finite non-zero jet widths at infinity in the z-plane, $f(w)$ should have logarithmic singularities at A_1, A_3, A_3', A_5. At A_2, A_2', A_4, A_4', and A_6 the function $f(w)$ should be finite, and as a matter of convenience, arbitrarily choose

$$f(0) = 0. \tag{2.9}$$

3. Conformal mapping of hodograph image onto half-plane. As an aid to determining the functional form of $w^2f'(w)$ it will be convenient to map the interior of the curve shown in Fig. 2, slit along A_5A_6, onto a half plane. To do this, first note that

$$W = \log (w/U_1) \tag{3.1}$$

maps the region in question onto the shaded region with polygonal boundary shown in Fig. 3. If we take account of symmetry, then by the Schwarz-Christoffel formula,

$$W = -\alpha \int_0^\zeta \frac{(\xi^2 - a_4^2)\, d\xi}{[(\xi^2 - 1)(\xi^2 - a_3^2)(\xi^2 - a_5^2)]^{0.5}}, \tag{3.2}$$

where $1 \leqq a_3 \leqq a_4 \leqq a_5$ will, for suitable choices of the positive parameters α, a_3, a_4, and a_5 yield the desired mapping onto the upper half of the ζ-plane. In the calculation of (3.2) use that branch of the integrand that is positive for $\zeta > a_5$. For later reference we also note that

$$\frac{1}{w}\frac{dw}{d\zeta} = -\frac{\alpha(\zeta^2 - a_4^2)}{[(\zeta^2 - 1)(\zeta^2 - a_3^2)(\zeta^2 - a_5^2)]^{0.5}}, \tag{3.3}$$

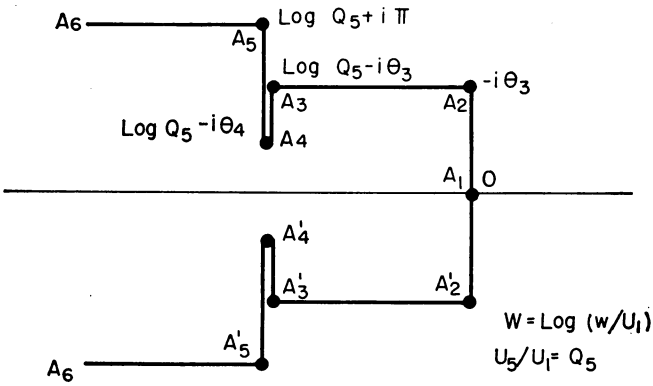

FIG. 3. Map in W plane.

$$\alpha I_1 = \alpha \int_0^1 \frac{(a_4^2 - \xi^2)\, d\xi}{[(1 - \xi^2)(a_3^2 - \xi^2)(a_5^2 - \xi^2)]^{0.5}} = -\theta_3 , \tag{3.4}$$

$$\alpha I_2 = \alpha \int_1^{a_3} \frac{(a_4^2 - \xi^2)\, d\xi}{[(\xi^2 - 1)(a_3^2 - \xi^2)(a_5^2 - \xi^2)]^{0.5}} = \log U_1/U_5 , \tag{3.5}$$

$$\alpha I_3 = \alpha \int_{a_3}^{a_4} \frac{(\xi^2 - a_4^2)\, d\xi}{[(\xi^2 - 1)(\xi^2 - a_3^2)(a_5^2 - \xi^2)]^{0.5}} = \theta_3 - \theta_4 , \tag{3.6}$$

$$\alpha I_4 = \alpha \int_{a_4}^{a_5} \frac{(\xi^2 - a_4^2)\, d\xi}{[(\xi^2 - 1)(\xi^2 - a_3^2)(a_5^2 - \xi^2)]^{0.5}} = \pi + \theta_4 . \tag{3.7}$$

The constant α can be evaluated as follows. Since the only singularities of $dW/d\zeta$ are at ± 1, $\pm a_3$, and $\pm a_5$, then

$$-4\alpha i(I_1 + I_3 + I_4) = \int_{C_1+C_2+C_3} (dW/d\xi)\, d\xi = \int_C (dW/d\xi)\, d\xi$$

where C_1, C_2, and C_3 are paths shown schematically in Fig. 4, and C is a circle $|\xi| = \text{const} > a_5$. Clearly

$$\int_C (dW/d\xi)\, d\xi = -\alpha \int_0^{2\pi} (\zeta^{-1} + \cdots)\, d\zeta = -2\pi i\alpha.$$

Since by (3.4) to (3.7), $-4\alpha i(I_1 + I_3 + I_4) = -4\pi i$, this yields

$$\alpha = 2. \tag{3.8}$$

4. Construction of $f(w)$. Recall that $w^2 f'(w)$ is alternately real or pure imaginary on the segments of the real axis of the ζ plane with end points A_γ and A'_γ, for $\gamma = 2, 3, 5$. Thus it must have branch points at these places, and should contain a factor

$$(\zeta^2 - 1)^{r/2}(\zeta^2 - a_3^2)^{s/2}(\zeta^2 - a_5^2)^{t/2}$$

where r, s, and t are odd integers. Since f should have logarithmic singularities at A_1, A_δ and A'_δ, $\delta = 3, 5$, then $df/d\zeta$ should have simple poles at the corresponding points. Since furthermore f should be finite at A_2 and A'_2 this suggests the form

$$w^2 f'(w) = \frac{\beta(\zeta^2 - 1)^{0.5}}{\zeta(\zeta^2 - a_3^2)^{0.5}(\zeta^2 - a_4^2)(\zeta^2 - a_5^2)^{0.5}}. \tag{4.1}$$

The factor $\zeta^2 - a_4^2$ in the denominator will enable us to include the case $a_4 = a_3$ in the following discussion. Now, by (4.1) and (3.3)

$$\frac{df}{d\zeta} = \frac{-2\beta}{w} \cdot \frac{1}{\zeta(\zeta^2 - a_3^2)(\zeta^2 - a_5^2)}, \tag{4.2}$$

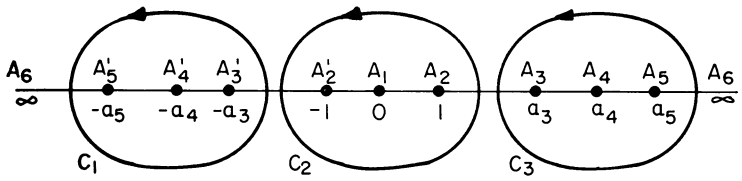

FIG. 4. Map in ζ plane.

where $\beta > 0$ merely determines the geometrical scale in the $z=f(\zeta)$ plane. Since $w(\infty)=0$, then by (2.9) $f(w(\infty)) = f(0) = 0$, and

$$f(\zeta) = \int_{\infty}^{\zeta} (df/d\zeta)\, d\xi. \tag{4.3}$$

The uniqueness of our choice for (4.1) can be shown as follows. Let us multiply the left members of (4.1) and (4.2) by an analytic function $H(\zeta)$. To preserve the alternation of real and imaginary values of $w^2f'(w)$ on the real axis, H must be real there. To prevent the introduction of new branch points and singularities, H must have no singularities in the closed upper-half plane, and the analytical continuation of H into the lower half plane is also free of singularities. Hence H is constant.

It remains to show that $f(w)$ has the desired properties at A_1, A_6, A_γ, and A'_γ. First note that $\zeta = 0$ corresponds to $w = U_1$. Thus by (3.1) and (3.2)

$$\zeta = (w - U_1)g(w - U_1),$$

where g is an analytic function of $w - U_1$ and $g(0) \neq 0$. Since $df/d\zeta$ has a simple pole at $\zeta = 0$, then $f(w(\zeta))$ has a logarithmic singularity there, and thus $f(w)$ also has a logarithmic singularity at $w = U_1$. A similar argument determines the behavior of f at $\zeta = \pm a_3$ or $\pm a_5$, with the unimportant difference that, for example, (3.1) and (3.2) imply

$$(\zeta - a_3)^{0.5} = (w - U_5e^{-i\theta_3})h(w - U_5e^{-i\theta_3}),$$

where h is analytic, $h(0) \neq 0$, etc. Hence $f(w)$ has the required logarithmic singularities.

By (3.2), in the neighborhood of infinity

$$\frac{dW}{d\zeta} = -2 \sum_{1}^{\infty} c_n\zeta^{-n},$$

where $c_1 = 1$. Hence $W = -2 \log \zeta + m(1/\zeta)$ where m is an analytic function of $1/\zeta$. Then by (3.1)

$$w = \zeta^{-2}n(1/\zeta),$$

where n is analytic and $n(0) \neq 0$. Then by (4.2)

$$df/d\zeta = \zeta^{2-5}k(1/\zeta),$$

where k is analytic and $k(0) \neq 0$. Thus $f(\xi)$ will be regular at ∞.

The width of the jet at A_1 in the z-plane can be determined by considering the expansions

$$\frac{df}{d\zeta} = -\frac{2\beta}{U_1a_3^2a_5^2\zeta} + \cdots,$$

$$f = c_1 - \frac{2\beta}{U_1a_3^2a_5^2} \log \zeta + \cdots.$$

Then the jump in Im $\log \zeta$ at $\zeta = 0$ yields for the width at A_1

$$h_1 = 2\beta\pi/U_1a_3^2a_5^2 \tag{4.4}$$

and rate of mass flow

$$M_1 = 2\beta\pi\rho/a_3^2a_5^2. \tag{4.5}$$

Similarly, at A_3 or A_3' we have widths

$$h_3 = \beta\pi/U_5 a_3^2(a_5^2 - a_3^2) \tag{4.6}$$

and rate of mass flow

$$M_3 = \beta\pi\rho/a_3^2(a_5^2 - a_3^2), \tag{4.7}$$

and at A_5 and A_5' the total width

$$h_5 = 2\beta\pi/U_1 a_5^2(a_5^2 - a_3^2), \tag{4.8}$$

and

$$M_5 = 2\beta\pi\rho/a_5^2(a_5^2 - a_3^2) \tag{4.9}$$

As we would expect from the law of conservation of mass

$$M_1 + M_5 = 2M_3 .$$

Finally, the ratio of the rate of mass flow at A_1 to that at A_5 is

$$M_1/M_5 = a_5^2/a_3^2 - 1. \tag{4.10}$$

In Fig. 1 the straight walls were adjacent to the low pressure regions. Would it be possible to place them adjacent to the high pressure regions? If we proceed purely formally, we merely have to replace the simple poles of $df/d\zeta$ at $\pm a_3$ by simple poles at ± 1. However, the following intuitive considerations show that this process leads at least to an indeterminancy. Suppose a flow of the desired type exists. For our modification consider

$$d \log (df/d\zeta)/d\zeta = -w'/w + \cdots .$$

Near the end of one wall the behavior of this logarithmic derivative is dominated by the branch point at $\xi = a_3$. But for $a_3 < \xi < a_4$, $Iw'/w < 0$. Thus, as one would expect, the streamline leaving the wall at A_3 bends initially toward the low pressure region, as shown in Fig. 5. Now, without changing the flow field, we can extend the walls into the stagnant high pressure regions. Since by appropriate changes of scale we can always make the gap between the ends of the extended walls be of unit length, this means that the location of the point of detachment A_3 is indeterminate.

5. Partially channelled impinging jets. To produce flows with impinging jets that are partially bounded by straight walls on both sides, it will suffice to replace the simple

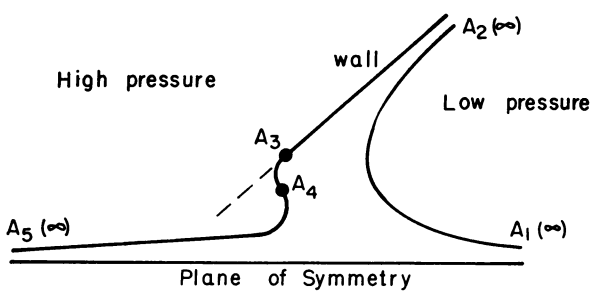

FIG. 5. Incoming jet with wall on high pressure side.

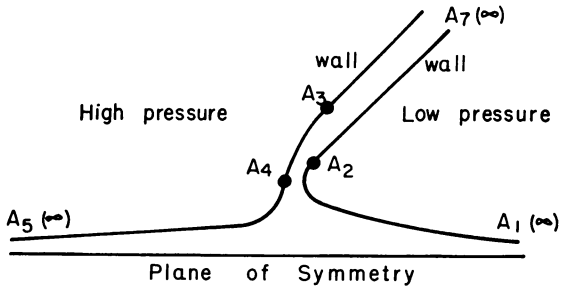

FIG. 6. Partially channelled incoming jet.

poles of $df/d\zeta$ at $\pm a_3$ by simple poles at $\pm a_7$, where $1 < a_7 < a_3$. Now (4.2) becomes

$$\frac{df}{d\zeta} = -\frac{2\beta}{w} \cdot \frac{1}{\zeta(\zeta^2 - a_7^2)(\zeta^2 - a_5^2)}, \tag{5.1}$$

while (3.2) remains unchanged. The presence of the additional parameter a_7 will make it possible to vary the location of A_3 in Fig. 6, for example, while the locations of A_2 and A_4 and the directions of the walls are held constant. The calculation of the various jet widths and rates of mass flow are perfectly straightforward exercises which we shall not repeat. It should be remarked that if there is an inflection point A_4 the wall A_7A_3 can be extended into the high pressure region again, just as in the discussion of Fig. 5.

SOME PROPERTIES OF INFINITE LINES*

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The transmission line equations in the lossless case may be written in the form

$$\frac{du(x)}{dx} + p(x)u(x) - i\omega v(x) = 0, \tag{1a}$$

$$\frac{dv(x)}{dx} - p(x)v(x) - i\omega u(x) = 0, \tag{1b}$$

where

$$p(x) = \frac{1}{2Z_0(x)} \frac{dZ_0(x)}{dx}, \tag{2}$$

and the characteristic impedance $Z_0(x)$ is real. In Ref. [1], solutions of Eqs. (1a) and (1b) were considered which have the asymptotic behavior,

$$\lim_{x\to\infty} u(x, \omega) \exp(-i\omega x) = 1 \tag{3a}$$

$$\lim_{x\to\infty} v(x, \omega) \exp(-i\omega x) = 1, \tag{3b}$$

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