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A THEORY OF NONLINEAR NETWORKS—I*

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Abstract. This report describes a new approach to nonlinear RLC-networks which is based on the fact that the system of differential equations for such networks has the special form

$$L(i)\frac{di}{dt} = \frac{\partial P(i, v)}{\partial i}, \qquad C(v)\frac{dv}{dt} = -\frac{\partial P(i, v)}{\partial v}.$$

The function, $P(i, v)$, called the mixed potential function, can be used to construct Liapounov-type functions to prove stability under certain conditions. Several theorems on the stability of circuits are derived and examples are given to illustrate the results. A procedure is given to construct the mixed potential function directly from the circuit. The concepts of a complete set of mixed variables and a complete circuit are defined.

Introduction. A. In the extensive theory of electrical circuits many impressive advances have led to a powerful tool for the engineer and the designer. For a wide class of problems one is able to construct a circuit with required properties using a rather complete theory which is available in several textbooks (see, e.g., [1], [2]). Most of these theories are based on the linear differential equations of electrical circuits. However, in recent times many engineering problems have led to the study of nonlinear networks which cannot appropriately be approximated by linear equations. Typical examples in this direction are the so-called flip-flop circuits which have several equilibrium states. Since a linear circuit obviously admits only one equilibrium, a flip-flop circuit can only be described by nonlinear differential equations. The main difference between such circuits and linear ones lies in the nonmonotone character of the voltage-current relations for the resistors. It will be a main point in the following to admit such "negative resistors".

B. The electrical circuits considered in this paper are general RLC-circuits in which any or all of the elements may be nonlinear. One of the purposes of this paper is to show that the differential equations of such electrical circuits have a special form which has its ultimate basis in the conservation laws of Kirchhoff. It will be derived that under very general assumptions the differential equations have the form

$$\begin{aligned} L_\rho \frac{di_\rho}{dt} &= \frac{\partial P}{\partial i_\rho}, \qquad (\rho = 1, \cdots, r), \\ C_\sigma \frac{dv_\sigma}{dt} &= -\frac{\partial P}{\partial v_\sigma}, \qquad (\sigma = r + 1, \cdots, r + s), \end{aligned} \tag{1}$$

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where the i_ρ represent the currents in the inductors and v_σ the voltages across the capacitors. The function $P(i, v)$ describes the physical properties of the resistive part of the circuit. Since it has the dimension of voltage times current, it will be called a potential function. This function can be formed additively from potential functions of the single elements similar to the way that the Hamiltonian is formed in particle dynamics from the potential energy and the kinetic energy of the different particles. However, it should be observed that equations (1) do not represent a Hamiltonian system since the latter describes nondissipative motion while in equations (1) the potential P contains dissipative terms. Also, the transformation properties of the above equations are different from Hamiltonian equations in that equations (1) preserve their form under coordinate transformations which leave the indefinite metric

$$-\sum_{\rho=1}^{r} L_\rho(di_\rho)^2 + \sum_{\sigma=1+r}^{s+r} C_\sigma(dv_\sigma)^2 \tag{2}$$

invariant.

C. A geometrical interpretation of the special form of equations (1) is the following. We consider a box containing an electrical circuit with only resistive elements. There are n pairs of terminals on the box which are connected internally to the electrical circuit. To measure the external electrical properties of this box we connect each terminal to either a current source of prescribed current i_ρ ($\rho = 1, \cdots, r$) or a voltage source of prescribed voltage v_σ ($\sigma = r + 1, \cdots, r + s = n$). Under natural compatibility assumptions for the arrangement of these sources, an equilibrium state (i_ν, v_ν) ($\nu = 1, \cdots, n$) will be attained, i.e., the missing quantities v_ρ ($\rho = 1, \cdots, r$) and i_σ ($\sigma = r + 1, \cdots, r + s$) will be determined. In other words the $2n$ voltages and currents satisfy n relations which define an n-dimensional surface in $2n$-dimensional space. We call this surface the characteristic surface Σ of the box. In fact, if $n = 1$, Σ is a curve usually called the voltage-current characteristic for an element or a circuit. The result that the equations have the form (1) can be expressed compactly by the identity,

$$\sum_{\nu=1}^{n} di_\nu \wedge dv_\nu = 0, \tag{3}$$

i.e., this two-dimensional differential form in the sense of Cartan [3] vanishes identically on the surface Σ. This fact will be explained and proved in section 13, part II.

D. It is also the purpose of this study to draw some conclusions concerning the solutions of the differential equations (1) from their special form. To show that such implications can be expected, consider, for instance, an RC-circuit (i.e., a circuit without inductors or $r = 0$ in (1)). In this case, the quadratic form (2) is positive definite and can be used as a metric

$$(ds)^2 = \sum_{\sigma=1}^{s} C_\sigma(dv_\sigma)^2.$$

One verifies immediately that in this case $P(i, v)$ decreases along solutions of (1) since

$$\frac{dP}{dt} = \sum_{\sigma=1}^{s} \frac{\partial P}{\partial v_\sigma}\frac{dv_\sigma}{dt} = -\left(\frac{ds}{dt}\right)^2$$

which is negative except at the equilibrium points. This implies that all solutions of an RC-circuit approach equilibrium states for $t \to \infty$ even if the resistors are negative in some regions. Of course, some natural assumptions have to be added and these will be found in section 8.

Especially in case a circuit contains negative resistors is it of interest to find criteria which guarantee that the solutions approach the equilibria as time increases and therefore do not oscillate. We saw that this is generally the case for RC-circuits and similarly for RL-circuits. On the other hand, RLC-circuits certainly will admit oscillations in general even in the linear case. But one would expect a nonoscillatory behavior of circuits in which the inductance—or a quantity of the dimension L/R^2C—is sufficiently small. Such criteria for nonoscillation will be derived in section 8. The main idea is to associate with the differential equation another metric which is positive and so find a function P^* which decreases along the solutions.

Such criteria are especially valuable for large circuits which contain many loops. It is usually hard to judge intuitively whether the presence of many loops may lead to oscillatory behavior. In section 9 we discuss an example of an arbitrarily large ladder network containing nonlinear elements, which demonstrates that our criteria are the best possible in general.

In section 20, part II, similar methods are used to establish the existence of periodic solutions for periodically excited nonlinear circuits. This result can be considered as an extension of a theorem of R. Duffin [4].

This paper is divided into two parts. The more important part is the first which leads to the main results rather directly without containing all the detailed proofs and refinements. The second part contains several additional results as well as detailed proofs complementing part I.

Originally, this work started with the study of some nonlinear circuits proposed by Goto and others [5]. Some preliminary investigations in this direction have been published earlier (see [6, 7]). In this paper, we present these ideas in a more systematic fashion in the hope that it will be useful to the theoretically inclined electrical engineers as well as mathematicians.

1. Complete sets of variables for a network. A network is an idealized concept in circuit theory which can be defined as a set of points, called nodes, and a set of connecting lines, called branches. It is irrelevant whether nodes and branches lie in a plane or whether the branches can be realized by straight lines. It is essential, however, that every branch connect exactly two nodes. Such a network is frequently called a graph. Actually, for applications other natural restrictions—like connectedness of the graph—could be imposed which, however, we will not need.

In each branch labeled by $\mu = 1, \cdots, b$ we specify a direction arbitrarily, indicating it by an arrow ("directed" graph). Accordingly, we distinguish the two connected nodes as initial and end nodes. The current flow in such a network is completely described by giving the amount of current i_μ flowing in the direction of the arrow; that means i_μ is negative if the flow is against the specified direction and positive otherwise. Similarly, we associate with each branch a voltage v_μ with a specified sign by taking the voltage level at the end node minus the voltage level at the initial node of the branch.

The $2b$ variables i_μ, v_μ ($\mu = 1, \cdots, b$) are restricted by the well-known Kirchhoff laws. The node law expresses that the currents arriving at any node (taken with proper sign) add up to zero which we write symbolically in the form

$$\sum_{\text{node}} \pm i_\mu = 0. \tag{1.1}$$

Kirchhoff's loop law expresses that the voltage drop over any loop (closed chain of branches) is zero, or

$$\sum_{\text{loop}} \pm v_\mu = 0. \tag{1.2}$$

Another way of describing this loop law is that to every node one can assign a voltage level such that v_μ equals the difference of the defined voltage levels between the end node and initial node.*

In the investigation of the circuit dynamics it will be of first importance to know how restrictive Kirchhoff's laws are. They form a set of linear equations, and we study first which of the currents and voltages can be chosen independently. More precisely, we call a set of variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$** "*complete*" if they can be chosen *independently* without leading to a violation of Kirchhoff's laws and if they determine in each branch *at least one* of the two variables, the current or the voltage. The problem is to describe a complete set of variables for a given graph.

This can be done in several ways. For instance, for $r = 0$ the answer is well-known. Choose in the graph a maximal tree τ, i.e., a subgraph which does not contain any loops and cannot be enlarged as a tree. It is clear that the t voltages, $v_1, \cdots, v_t$, on this tree can be assigned arbitrarily without interfering with Kirchhoff's loop law since the tree does not contain any loops. The branches not contained in the tree are called "links", and it remains to determine the voltages in the links. Since τ is maximal, such a link added to τ forms a loop, and therefore the link voltage is expressible in terms of the tree voltages $v_1, \cdots, v_t$ by (1.2). This proves that *the voltages in a maximal tree form a complete set of variables*, $(r = 0, s = t)$.

Similarly, it is well-known how to choose a complete set of currents. Let τ be a maximal tree and $\mathfrak{L}$ its l links. Decomposing the graph into independent loops, one finds readily that *the link currents of a maximal tree form a complete set of variables*, $(r = l, s = 0)$. See Guillemin [2].

Finally, we construct a complete set of variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ in the "mixed" case where $rs > 0$. For this purpose we begin with a maximal tree τ (with a corresponding set of links $\mathfrak{L}$) and choose a subtree τ' in τ. With $\mathfrak{L}'$ (links of τ') we denote all branches which connect two nodes of τ' and make a loop with branches of τ' only.† Thus $\mathfrak{L}'$ is contained in $\mathfrak{L}$ and τ' is a maximal tree of the graph $\tau' + \mathfrak{L}'$. The number of branches in τ', $\mathfrak{L}'$ will be denoted by t', l' respectively. Then, *the currents* $i_1, \cdots, i_r$ *in the branches of* $\mathfrak{L} - \mathfrak{L}'$ *together with the voltages* $v_{r+1}, \cdots, v_{r+s}$ *in* τ' *form a complete set of variables*, $(r = l - l', s = t')$.

This is easily seen. Since τ' is a maximal tree of $\tau' + \mathfrak{L}'$, the voltages $v_{r+1}, \cdots, v_{r+s}$ are independent and determine all voltages in $\tau' + \mathfrak{L}'$. Also, the currents $i_1, \cdots, i_r$—being link currents—are independent, and it remains to be shown that the currents in all branches outside $\tau' + \mathfrak{L}'$ are determined too. We form the r loops through the links $\mathfrak{L} - \mathfrak{L}'$ whose loop currents are $i_1, \cdots, i_r$. Recalling that *all* l loop currents determine all branch currents and the fact that the loops through the l' links of $\mathfrak{L}'$ belong entirely

*A more analytic description of these laws with a connection matrix of the graph will be found in section 12, part II.

**We always reserve the freedom to relabel the branches and choose here the branches corresponding to the given set of variables as the first ones.

†It could happen that τ' is not connected and a branch connecting two nodes in different components of τ' would not form a loop within τ'. These branches are to be excluded from $\mathfrak{L}'$.

to $\tau' + \mathfrak{L}'$, it follows that all branch currents outside $\tau' + \mathfrak{L}'$ are independent of the l' loop currents and hence are already determined by the $r = l - l'$ loop currents $i_1, \cdots, i_r$.

One sees that the mixed case of a complete set contains, in particular, the "pure" cases. If τ' is empty, one has $s = 0$ and $r = l$, i.e., the case of a pure set of currents; if $\tau' = \tau$, one has $r = l - l' = 0$, $s = t$, which gives a pure set of t voltages.

We remark that the link currents in $\mathfrak{L}'$ and the voltages in $\tau - \tau'$ give rise to another complete set of variables.

In the mixed case the whole graph $\mathfrak{N}$ is broken up into $\mathfrak{N}_v = \tau' + \mathfrak{L}'$, those branches in which the voltages can be determined from $v_{r+1}, \cdots, v_{r+s}$ by Kirchhoff's loop law, and the remaining branches $\mathfrak{N}_i$ in which the currents can be determined from $i_1, \cdots, i_r$ by Kirchhoff's node law.

2. Network theorems. The two theorems in this section are derived using only the geometry of the network and Kirchhoff's laws. They do not depend on the types of elements in the branches. Theorem 1 is known as Tellegen's theorem [8], but it is stated here in a form which has a geometrical interpretation.

We consider a directed network with b branches and n nodes. The set of branch currents $i = (i_1, \cdots, i_b)$ and the set of branch voltages $v = (v_1, \cdots, v_b)$ are vectors in the b-dimensional Euclidean vector space $\mathcal{E}_b$. The inner product is defined for two vectors $x, y \in \mathcal{E}_b$ as $(x, y) = \sum_{\mu=1}^{b} x_\mu y_\mu$. Let $\mathcal{I}$ be the set of all vectors in $\mathcal{E}_b$ such that if $x \in \mathcal{I}$ and the components of x are taken as the branch currents of the directed network, then Kirchhoff's node law, $\sum_{\text{node}} \pm x_\mu = 0$, must hold at every node. Similarly, we let $\mathcal{V}$ denote the set of all vectors in $\mathcal{E}_b$ such that if $x \in \mathcal{V}$ and the components of x are taken as the branch voltages, then Kirchhoff's loop law, $\sum_{\text{loop}} \pm x_\mu = 0$, should be satisfied for every loop. It is obvious that $\mathcal{I}$ and $\mathcal{V}$ are subspaces of $\mathcal{E}_b$ since they are defined through linear equations.

Theorem 1. If $i \in \mathcal{I}$ and $v \in \mathcal{V}$, then $(i, v) = 0$, i.e., $\mathcal{I}$ and $\mathcal{V}$ are orthogonal subspaces of $\mathcal{E}_b$.

Proof. Since v satisfies Kirchhoff's loop law, there exists a set of node voltages $(V_1, \cdots, V_n)$ such that v_μ is the difference between the voltages of the end node and the initial node (see previous section). Let the current flowing from node k to node l be denoted by i_{kl} which is taken to be zero if there is no connecting branch. Thus, if the μth branch connects nodes k and l, we have

$$v_\mu i_\mu = (V_k - V_l) i_{kl} = (V_l - V_k) i_{lk}^{*},$$

and because of this symmetry in k and l, we can express the inner product (i, v) as a free sum, i.e.,

$$(i, v) = \sum_{\mu=1}^{b} i_\mu v_\mu = \tfrac{1}{2} \sum_{k,l} (V_k - V_l) i_{kl},$$

or

$$(i, v) = \tfrac{1}{2}\Big[\sum_k V_k \Big(\sum_l i_{kl}\Big) - \sum_l V_l \Big(\sum_k i_{kl}\Big)\Big].$$

However,

$$\sum_l i_{kl} = \sum_{\text{node } k} \pm i_\mu = 0 \quad \text{and} \quad \sum_k i_{kl} = \sum_{\text{node } l} \pm i_\mu = 0$$

so that $(i, v) = 0$.

*There is no restriction in assuming that at most one branch connects the same two nodes.

We remark that $\mathcal{J}$ and $\mathcal{V}$ are not only orthogonal subspaces, but that they even span $\mathcal{E}_b$. The simple proof of this fact is found in section 12, part II, although it is used in the next theorem.

The next theorem is similar but not equivalent to the first theorem and leads directly to one of the main results of this paper. Let Γ denote a one-dimensional curve in $\mathcal{E}_b$ with projections on $\mathcal{J}$ and $\mathcal{V}$ denoted by i and v, respectively.

Theorem 2.

$$\int_\Gamma \sum_{\mu=1}^{b} v_\mu \, di_\mu = \int_\Gamma \sum_{\mu=1}^{b} i_\mu \, dv_\mu = 0.$$

Proof. Since $di = (di_1, \cdots, di_b)$ is the limit of the difference of two vectors in $\mathcal{J}$ and since $\mathcal{J}$ is a subspace, then $di \in \mathcal{J}$ and by theorem 1

$$(v, di) = \sum_{\mu=1}^{b} v_\mu \, di_\mu = 0.$$

Integrating along Γ we obtain

$$\int_\Gamma \sum_{\mu=1}^{b} v_\mu \, di_\mu = 0,$$

and integration by parts yields

$$\int_\Gamma \sum_{\mu=1}^{b} v_\mu \, di_\mu = (i, v) \Big|_\Gamma - \int_\Gamma \sum_{\mu=1}^{b} i_\mu \, dv_\mu .$$

Since $(i, v) = 0$ by theorem 1, we have

$$\int_\Gamma \sum_{\mu=1}^{b} i_\mu \, dv_\mu = 0.$$

3. Nonlinear elements. So far we have only discussed facts which depend on network concepts, and we have seen that Kirchhoff's laws impose certain restrictions among the branch voltages and among the branch currents. On the other hand, physical properties of the elements in the branches lead to further restrictions which relate branch currents to branch voltages, and it is our purpose here to discuss the nature of these relations.

We consider elements which are two terminal devices and restrict our discussion to purely resistive, inductive or capacitive elements.

The name "resistor" usually refers to a linear passive device which has a resistance R such that the current and voltage at its terminals are related by $v = -Ri$.* A more general concept is obtained by considering a resistor as a continuous function such that the relation $f(i, v) = 0$ holds. This defines a continuous curve in the (i, v)-plane which we will call the characteristic of the resistor. From $f(i, v) = 0$ we could solve for i or v as a function of the other. It is not necessary to require that either such function be single valued, and, in fact, the more interesting cases are when this is not true. For example, the tunnel diode [9] is a nonlinear resistor which has the characteristic shown in figure 1, and clearly v as a function of i is not single valued.

It will also not be necessary to require that the characteristic pass through the origin,

*The negative sign is chosen here in order to be consistent with the convention adopted in Section 1 on the direction of positive current and positive voltage.

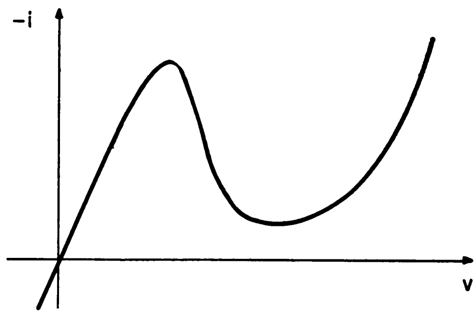

FIG. 1. Voltage-Current Characteristic for a Tunnel Diode

i.e., that the element be passive. A generator (voltage or current) is therefore a special type of resistor for which the characteristic is parallel to one of the axes.

There are some assumptions which will be made about resistors. It will be assumed that there exists $B > 0$ such that for $|i|, |v| > B$ the characteristic of the resistor lies in the first and third quadrants and is monotone increasing there. We make this assumption in order to guarantee that the equilibrium problem can be solved, i.e., that we can solve the circuit equations under steady state conditions. A proof of this statement under less stringent assumptions will be found in section 14, part II.

For completeness we discuss the well-known laws for inductors and capacitors which we also allow to be nonlinear. An inductor is a function relating the magnetic flux linkages to the current, i.e.,

$$\phi = -f(i).$$

In terms of voltage and current

$$v = \frac{d\phi}{dt} = -f'(i)\frac{di}{dt} = -L(i)\frac{di}{dt},$$

where $L(i)$ is the inductance and is non-negative. Similarly, a capacitor is a function relating the charge and the voltage, i.e.,

$$q = -f(v).$$

Differentiating, we obtain

$$i = \frac{dq}{dt} = -f'(v)\frac{dv}{dt} = -C(v)\frac{dv}{dt},$$

where $C(v)$ is the capacitance and is non-negative.

We remark that mutual inductance can be handled by simply changing to vector notation where $L(i)$ would be a symmetric matrix.

4. The form of the equations. The general RLC circuit can be thought of as a resistive circuit (R-circuit) with n ports to which either an inductor or a capacitor is attached (Fig. 2). We want to derive the differential equations describing the dynamical behavior of such a circuit.

We assume that the resistors are of the type discussed in the previous section so that the equilibrium problem can be solved. This means that if we know the currents denoted by $i^* = (i_1, \cdots, i_r)$ in all the inductors and the voltages denoted by $v^* = (v_{r+1}, \cdots, v_{r+s})$

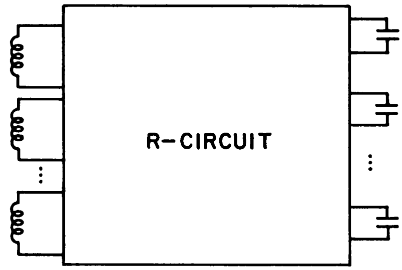

FIG. 2. General RLC Circuit

across all the capacitors, the problem of determining the voltage and current in any branch of the circuit can be solved from some implicit equations, i.e.,

$$\begin{aligned} v_\mu &= f_\mu(i^*, v^*), \qquad \mu = 1, \cdots, b, \\ i_\mu &= g_\mu(i^*, v^*), \qquad \mu = 1, \cdots, b. \end{aligned} \tag{4.1}$$

In general, there may be several solutions for v_μ, i_μ leading to multiple-valued functions f_μ, g_μ with branch points, cusps, etc.

Of particular interest is the voltage across any inductor and the current through any capacitor given by

$$\begin{aligned} v_\rho &= f_\rho(i^*, v^*), \qquad \rho = 1, \cdots, r, \\ i_\sigma &= g_\sigma(i^*, v^*), \qquad \sigma = r + 1, \cdots, r + s. \end{aligned} \tag{4.2}$$

On the other hand, the dynamical laws of inductors and capacitors, as discussed in section 3, require

$$\begin{aligned} v_\rho &= -L_\rho(i_\rho)\frac{di_\rho}{dt}, \qquad \rho = 1, \cdots, r, \\ i_\sigma &= -C_\sigma(v_\sigma)\frac{dv_\sigma}{dt}, \qquad \sigma = r + 1, \cdots, r + s. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we obtain the desired differential equations,

$$\begin{aligned} L_\rho(i_\rho)\frac{di_\rho}{dt} &= -f_\rho(i^*, v^*), \qquad \rho = 1, \cdots, r, \\ C_\sigma(v_\sigma)\frac{dv_\sigma}{dt} &= -g_\sigma(i^*, v^*), \qquad \sigma = r + 1, \cdots, r + s \end{aligned} \tag{4.4}$$

which give di^*/dt and dv^*/dt explicitly in terms of i^* and v^*.

It is our aim to show that the functions on the right-hand side of (4.4) can be derived from one function.

For this purpose we make use of theorem 2

$$\int_\Gamma \sum_{\mu=1}^{b} v_\mu \, di_\mu = 0, \tag{4.5}$$

where μ ranges over *all* branches. For Γ we will now choose any curve from a fixed state to a variable one in such a manner that along Γ the voltages and currents satisfy the relations characteristic for the resistors. This means we choose $i^*(s)$, $v^*(s)$ arbitrarily as continuous functions and determine the remaining components i_μ, v_μ for $\mu > r + s$ and v_ρ, i_σ ($\rho = 1, \cdots, r; \sigma = r + 1, \cdots, r + s$) from the relations (4.1). Although the f_μ, g_μ are possibly multiple-valued, we make, by a particular choice, the $v_\mu(s)$, $i_\mu(s)$, $v_\rho(s)$, $i_\sigma(s)$ single-valued and continuous functions of s. This is, in general, possible if one excludes pathological functions, but we do not try to give a precise discussion of necessary restrictions and simply assume that such a choice of continuous functions exists. Since $\mathcal{J}$ and $\mathcal{V}$ span $\mathcal{E}_b$, this defines a curve Γ in $\mathcal{E}_b$ from a fixed point to a variable point determined by $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ only.

With Γ chosen in the manner specified, we now make the obvious but important observation that the integral,

$$\int_\Gamma \sum_{\mu > r+s} v_\mu \, di_\mu \, ,$$

taken *not* over all branches as in (4.5) but only over all resistor branches, depends only on the end points of Γ. This simply follows from the fact that in a resistor v_μ depends on i_μ only, i.e., that a resistor relates the current and voltage in one and the same branch only.

We write (4.5) in the form

$$\int_\Gamma \sum_{\rho=1}^{r} v_\rho \, di_\rho + \int_\Gamma \sum_{\sigma=r+1}^{r+s} v_\sigma \, di_\sigma + \int_\Gamma \sum_{\mu>r+s} v_\mu \, di_\mu = 0 \tag{4.6}$$

or, integrating the second line integral by parts,

$$\int_\Gamma \sum_{\rho=1}^{r} v_\rho \, di_\rho - \int_\Gamma \sum_{\sigma=r+1}^{r+s} i_\sigma \, dv_\sigma + P = 0, \tag{4.7}$$

where

$$P = \int_\Gamma \sum_{\mu>r+s} v_\mu \, di_\mu + \sum_{\sigma=r+1}^{r+s} i_\sigma v_\sigma \Big|_\Gamma \tag{4.8}$$

is a function depending only on the end points of Γ. In other words, P is a function of the variable end point of Γ which, in turn, depends only on the variables $i_1, \cdots, i_r$, $v_{r+1}, \cdots, v_{r+s}$, i.e., $P = P(i^*, v^*)$. It is also only defined up to a constant which depends on the choice of the fixed initial point of Γ. From (4.7) we read off

$$\begin{aligned} v_\rho &= -\frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r, \\ i_\sigma &= +\frac{\partial P}{\partial v_\sigma}, \qquad \sigma = r+1, \cdots, r+s \end{aligned} \tag{4.9}$$

which, with (4.2) and (4.4), gives the desired differential equations

$$\begin{aligned} L_\rho(i_\rho) \frac{di_\rho}{dt} &= \frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r, \\ C_\sigma(v_\sigma) \frac{dv_\sigma}{dt} &= -\frac{\partial P}{\partial v_\sigma}, \qquad \sigma = r+1, \cdots, r+s. \end{aligned} \tag{4.10}$$

These equations put into evidence that the right-hand sides are derived from one single function P. The computation of P required solving implicit equations and therefore is still complicated. But, we will show in the next section how to construct and interpret P in simple cases.

An immediate consequence of the equation (4.10) is that for linear circuits the equilibrium equations can be written with a symmetric R (resistance) matrix. This fact is discussed in section 15, part II, and is closely related to the reciprocity theorem.

5. Construction of P. We want to describe in some detail how the function P, which we will call the *mixed potential,* can be constructed directly from the circuit. For this purpose, we use the definition of P according to (4.8),

$$P(i^*, v^*) = \int_\Gamma \sum_{\mu > r+s} v_\mu \, di_\mu + \sum_{\sigma=r+1}^{r+s} i_\sigma v_\sigma \Big|_\Gamma , \tag{5.1}$$

where $i^* = (i_1, \cdots, i_r)$ is the set of currents through the inductors and $v^* = (v_{r+1}, \cdots, v_{r+s})$ is the set of voltages across the capacitors. Since v_μ depends only on i_μ for $\mu > r + s$, P can be written as

$$P(i^*, v^*) = \sum_{\mu > r+s} \int_\Gamma v_\mu \, di_\mu + \sum_{\sigma=r+1}^{r+s} i_\sigma v_\sigma \Big|_\Gamma . \tag{5.2}$$

The integral $\int_\Gamma v_\mu \, di_\mu$ is, of course, well-defined as a line integral even if v_μ cannot be written as a single-valued function of i_μ. Taken as a line integral, the path of integration is along the characteristic of the resistor, as can be easily seen from the definition of Γ. We give this integral a special name, the *current potential* of the element in the branch labeled by μ. Similarly, the line integral $\int_\Gamma i_\mu \, dv_\mu$ will be called the *voltage potential,* and it is easily seen that

$$\int_\Gamma i_\mu \, dv_\mu + \int_\Gamma v_\mu \, di_\mu = i_\mu v_\mu \Big|_\Gamma .^* \tag{5.3}$$

The current or voltage potential has a simple interpretation if the graph of a resistor can be expressed as a single-valued function of one of the variables. For example, if i_μ is a single-valued function of v_μ, then the voltage potential is an ordinary integral and consequently is the shaded area shown in figure 3 assuming that the initial fixed point of the path was at $v_\mu = 0$.

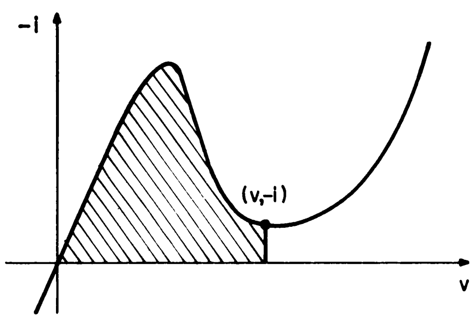

Fig. 3. Voltage Potential

*The current and voltage potential have been defined by W. Millar [10] and C. Cherry [11] who call them the contents and cocontents respectively.

We shall speak of the current (voltage) potential of a network as the sum of the current (voltage) potentials of its resistors.

It is useful to make some remarks about the sign of a resistor characteristic. The current induced through a linear resistor R is in the direction of the voltage drop, i.e., the induced current flows from + to − (Fig. 4). Now suppose that the direction assigned

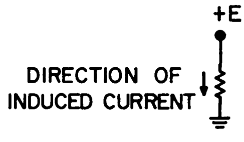

FIG. 4. Direction of Induced Current Through a Resistor

to this branch of the directed network is from E to ground. Then, according to the convention adopted in section 1 on the direction of positive current and voltage, the branch current i_μ is equal to i, the induced current, whereas the branch voltage $v_\mu = -E$. We find that $v_\mu = -Ri_\mu$ and the current potential is $-\frac{1}{2}Ri_\mu^2$. We obtain this even if the opposite direction were assigned to this branch because then $v_\mu = E$, $i_\mu = -i$, and again $v_\mu = -Ri_\mu$. For similar reasons, the current potential of a battery E is $+Ei_\mu$.

Now we may state the procedure for constructing the mixed potential directly from the circuit:

(1) determine the current potential for each resistor;

(2) determine the product $i_\sigma v_\sigma$ for each capacitor;

(3) form the sum of these terms and express it in terms of i^*, v^*.

We show now that the concept of a mixed potential contains the voltage potential and the current potential as special cases. If the circuit contains no capacitors, the second sum in the mixed potential is absent, and hence the mixed potential reduces to

$$P(i^*) = \int_\Gamma \sum_{\mu > r+s} v_\mu \, di_\mu ,$$

which is the current potential. In this case, the differential equations take the simple form

$$L_\rho \frac{di_\rho}{dt} = \frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r.$$

Similarly, if the circuit contains no inductors, the mixed potential contains a term for each branch, and by theorem 1, we have

$$P(v^*) = -\int_\Gamma \sum_{\sigma=1}^{s} i_\mu \, dv_\mu ,$$

i.e., $-P(v^*)$ is the voltage potential. The differential equations are

$$C_\sigma \frac{dv_\sigma}{dt} = -\frac{\partial P}{\partial v_\sigma}, \qquad \sigma = r+1, \cdots, r+s.$$

To illustrate the procedure for constructing the mixed potential, we consider a few examples.

Example 1. We consider the circuit shown in Fig. 5. The current potential of R is

$$\int_0^i (-Ri) \, di = -\tfrac{1}{2}Ri^2,$$

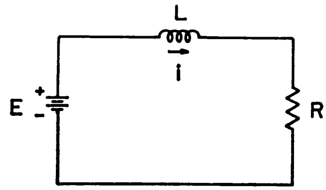

FIG. 5. Series RL Circuit

and the current potential of the battery is

$$\int_0^i E\,di = Ei.$$

Since no capacitors are present, we have

$$P(i) = Ei - \tfrac{1}{2}Ri^2,$$

which gives the desired differential equations by (4.10)

$$L\frac{di}{dt} = \frac{\partial P}{\partial i} = E - Ri.$$

Example 2. Next we consider the tunnel diode circuit shown in Fig. 6. The current

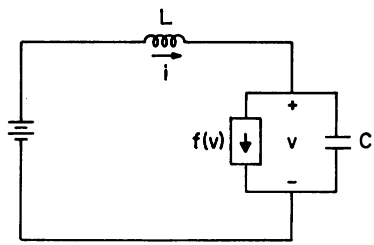

FIG. 6. Tunnel-Diode Circuit

through the box in the direction shown is given by the nonlinear function $f(v)$ shown in Fig. 1. According to (5.3) the current potential of the resistor is

$$-\int_\Gamma v\,d(f(v)) = -vf(v) + \int_0^v f(v)\,dv.$$

The current through the capacitor is $i - f(v)$ and, since it is in the opposite direction of v, the $i_\sigma v_\sigma$ product is $-(i - f(v))v$. Thus, the mixed potential is

$$P(i, v) = Ei + \int_0^v f(v)\,dv - iv,$$

which gives the differential equations

$$L\frac{di}{dt} = \frac{\partial P}{\partial i} = E - v,$$

$$C\frac{dv}{dt} = -\frac{\partial P}{\partial v} = i - f(v).$$

Of course, these examples are very simple, and, in general, it may be difficult although not impossible to express v_μ, i_μ in terms of the desired variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$. However, if these variables are complete, we know according to the discussion of section 1 that v_μ, i_μ can be simply expressed. Then the mixed potential function has a particularly simple and useful form which we discuss in the next section.

6. The mixed potential for complete circuits. We call a circuit *complete* if the set of variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ is complete, where $i_1, \cdots, i_r$ denote the currents through the inductors and $v_{r+1}, \cdots, v_{r+s}$ denote the voltages across the capacitors. It is our purpose here to discuss the form that the mixed potential takes for the class of complete circuits and to give a simple procedure for its construction.

Recall (Section 1) that a complete circuit could be split into two subnetworks $\mathfrak{N}_i$ and $\mathfrak{N}_v$ such that the branch currents of $\mathfrak{N}_i$ and the branch voltages of $\mathfrak{N}_v$ are known from the complete set of variables. In particular, $\mathfrak{N}_v$ contains all the capacitors and $\mathfrak{N}_i$ the inductors. According to (4.8), the mixed potential is*

$$\begin{aligned} P(i^*, v^*) &= \sum_{\mu > t} \int_\Gamma v_\mu \, di_\mu + \sum_{\sigma=r+1}^{t} v_\sigma i_\sigma \Big|_\Gamma \\ &= \sum_{\mu>t,\mathfrak{N}_i} \int_\Gamma v_\mu \, di_\mu + \sum_{\mu>t,\mathfrak{N}_v} \int_\Gamma v_\mu \, di_\mu + \sum_{\sigma=r+1}^{t} v_\sigma i_\sigma \Big|_\Gamma . \end{aligned} \tag{6.1}$$

The first term is simply the current potential of $\mathfrak{N}_i$ and, since i^* determines all the branch currents in $\mathfrak{N}_i$, we can express this term as a function of i^*, which we denote by $F(i^*)$. According to (5.3), the second term is

$$\sum_{\mu>t,\mathfrak{N}_v} \int_\Gamma v_\mu \, di_\mu = \sum_{\mu>t,\mathfrak{N}_v} \left[v_\mu i_\mu \Big|_\Gamma - \int_\Gamma i_\mu \, dv_\mu \right],$$

and therefore we have

$$P(i^*, v^*) = F(i^*) + \sum_{\mathfrak{N}_v} v_\mu i_\mu \Big|_\Gamma - \sum_{\mu>t,\mathfrak{N}_v} \int_\Gamma i_\mu \, dv_\mu .^{**}$$

The last term is simply the voltage potential of $\mathfrak{N}_v$ and, since v^* determines all branch voltages of $\mathfrak{N}_v$, we denote this by $G(v^*)$ and we have

$$P(i^*, v^*) = F(i^*) - G(v^*) + \sum_{\mathfrak{N}_v} v_\mu i_\mu \,|_\Gamma . \tag{6.2}$$

It remains to be shown that the last term of (6.2) can be expressed in terms of the complete set of variables $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ alone, which is not obvious since in every branch only one of the variables i_μ, v_μ is known.

Lemma. There exists an $r \times s$ matrix $\gamma = (\gamma_{\rho\sigma})$ with $\gamma_{\rho\sigma} = +1, -1, 0$ such that

$$\sum_{\mathfrak{N}_v} v_\mu i_\mu = \sum_{\rho=1,\sigma=1}^{r,s} \gamma_{\rho\sigma} i_\rho v_{r+\sigma} = (i^*, \gamma v^*). \tag{6.3}$$

Proof. We draw attention to the set $\mathfrak{N}_0$ of n_0 nodes at which branches of $\mathfrak{N}_i$ and $\mathfrak{N}_v$ come together. Since the currents in $\mathfrak{N}_i$ can be expressed in terms of $i_1, \cdots, i_r$, we can determine the currents j_ν, $(\nu = 1, \cdots, n_0)$, at the nodes of $\mathfrak{N}_0$ flowing from $\mathfrak{N}_i$ to $\mathfrak{N}_v$.

*For brevity we denote $r + s$ by t.

**Since the branches labeled by $\sigma = r + 1, \cdots, r + s$ containing the capacitors belong to $\mathfrak{N}_v$, the second term contains the last sum of (6.1).

From Kirchhoff's node law it follows that

$$\sum_{\nu=1}^{n_0} j_\nu = 0. \tag{6.4}$$

Similarly, since we can express the voltage difference between any two nodes of $\mathfrak{N}_0$ in terms of $v_{r+1}, \cdots, v_{r+s}$, we can assign a voltage level w_ν to every node in $\mathfrak{N}_0$. It is clear that the sum

$$\sum_{\nu=1}^{n_0} j_\nu w_\nu$$

can be expressed in terms of $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$. The proof will be complete if we identify this sum with the given one (up to the sign). For this purpose we dissect the graph $\mathfrak{N}$ at the nodes $\mathfrak{N}_0$ and replace $\mathfrak{N}_i$ by artificial branches from the first node to the νth node ($\nu = 2, \cdots, n_0$) (figure 7). We assign the currents j_ν and voltages $w_\nu - w_1$

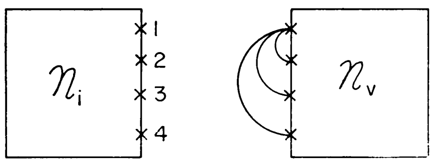

FIG. 7. The Network $\mathfrak{N}$ Dissected at the Nodes $\mathfrak{N}_0$

($\nu = 2, \cdots, n_0$) to these artificial branches. For the new graph consisting of the new branches and $\mathfrak{N}_v$, we have by theorem 1

$$\sum_{\mathfrak{N}_v} i_\mu v_\mu + \sum_{\nu=2}^{n_0} j_\nu(w_\nu - w_1) = 0 \tag{6.5}$$

or, by (6.4),

$$\sum_{\mathfrak{N}_v} i_\mu v_\mu = -\sum_{\nu=1}^{n_0} j_\nu w_\nu \,. \tag{6.6}$$

This shows that the sum on the left is expressed in terms of $j_1, \cdots, j_{n_0}, w_1, \cdots, w_{n_0}$ which are in turn expressible in terms of i^*, v^*. In fact, the j_ν depend linearly on i^* and the w_ν depend linearly on v^*. Hence, with some constants $\gamma_{\rho\sigma}$ we have

$$\sum_{\mathfrak{N}_v} i_\mu v_\mu = \sum \gamma_{\rho\sigma} i_\rho v_{r+\sigma} \,.$$

This proves the lemma except for the verification that $\gamma_{\rho\sigma} = \pm 1, 0$ which is left to section 13, part II.

Moreover, we have found an interpretation of the term $\sum_{\mathfrak{N}_v} i_\mu v_\mu$; it is the sum of the product of the current j_ν passing from $\mathfrak{N}_i$ to $\mathfrak{N}_v$ and the voltage level w_ν of the node summed over the nodes of $\mathfrak{N}_0$.

Combining (6.2) and (6.3) gives us the final form for the mixed potential in terms of the variables i^*, v^* only, i.e.,

$$P(i^*, v^*) = F(i^*) - G(v^*) + (i^*, \gamma v^*). \tag{6.7}$$

The above formula already gives us a procedure for constructing the mixed potential from a complete circuit; *the mixed potential is the sum of the current potential for* $\mathfrak{N}_i$, *the voltage potential for $\mathfrak{N}_v$, and a term $(i^*, \gamma v^*)$ which is constructed as outlined in the lemma.* We remark that our reason for calling $P(i^*, v^*)$ the *mixed* potential is a consequence of the form of (6.7).

There is a more direct and simpler method for constructing the cross-product term $(i^*, \gamma v^*)$ and we state this without proof. Take any link $\rho \in \mathfrak{L} - \mathfrak{L}'$ and consider the loop Λ_ρ which it determines. The branches of Λ_ρ other than the link branch are branches of the maximal tree τ of which some may be branches of τ'. The desired term is the sum of products of the loop current i_ρ and the voltage v_σ of the branches of τ' in Λ_ρ summed over all links of $\mathfrak{L} - \mathfrak{L}'$, i.e.,

$$(i^*, \gamma v^*) = \sum_{\rho=1}^{\tau} i_\rho \sum_{\Lambda_\rho \cap \tau'} \pm v_\sigma .$$

Now we consider a fairly complicated example of a complete circuit to illustrate the the procedure for constructing the mixed potential.

Example 3. Considered only as a graph, the circuit in Fig. 8 becomes the graph shown in Fig. 9, where the dots represent nodes. The maximal tree τ consists of branches $\{4, 5, 6, 9, 10, 11, 12, 13\}$, $\tau' = \{4, 5, 6\}$, $\mathfrak{L}' = \{7, 8\}$, $\mathfrak{L} - \mathfrak{L}' = \{1, 2, 3\}$ and we see

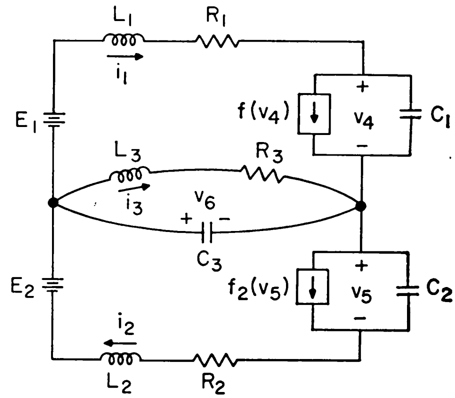

FIG. 8. Twin Tunnel-Diode Circuit

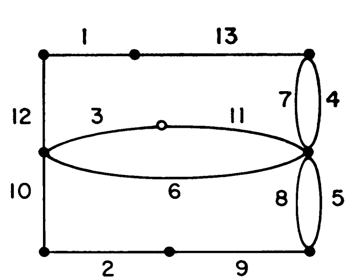

FIG. 9. The Graph of the Circuit of Figure 8

that i_1, i_2, i_3, v_4, v_5, v_6 form a complete set of mixed variables. We construct the current potential of $\mathfrak{N}_i = \{1, 2, 3, 9, 10, 11, 12, 13\}$ where terms come only from the resistive branches $\{9, 10, 11, 12, 13\}$ and are listed in order:

$$F(i_1, i_2, i_3) = -\tfrac{1}{2}R_2 i_2^2 + E_2 i_2 - \tfrac{1}{2}R_3 i_3^2 + E_1 i_1 - \tfrac{1}{2}R_1 i_1^2 . \tag{6.8}$$

Similarly, the current potential of $\mathfrak{N}_v = \{4, 5, 6, 7, 8\}$ is due to the resistive branches $\{7, 8\}$ and is

$$G(v_4, v_5, v_6) = -\int_0^{v_4} f_1(v)\,dv - \int_0^{v_5} f_2(v)\,dv. \tag{6.9}$$

It remains to determine $(i^*, \gamma v^*)$ which we do by considering the loops $\Lambda_1 = \{1, 13, 4, 6, 12\}$, $\Lambda_2 = \{2, 10, 6, 5, 9\}$ and $\Lambda_3 = \{3, 11, 6\}$ and their intersections with τ'. We obtain the correct sign by determining which way the loop currents i_ρ $(\rho = 1, 2, 3)$ flow through the voltages v_σ $(\sigma = 4, 5, 6)$ of τ' which gives us

$$(i^*, \gamma v^*) = i_1(-v_4 + v_6) + i_2(-v_6 - v_5) + i_3(v_6). \tag{6.10}$$

Combining (6.8), (6.9), and (6.10) according to (6.7), we obtain the mixed potential for this circuit, i.e.,

$$\begin{aligned} P(i^*, v^*) = {} & E_1 i_1 + E_2 i_2 - \tfrac{1}{2}R_1 i_1^2 - \tfrac{1}{2}R_2 i_2^2 - \tfrac{1}{2}R_3 i_3^2 \\ & + \int_0^{v_4} f_1(v)\,dv + \int_0^{v_5} f_2(v)\,dv - i_1 v_4 + i_1 v_6 - i_2 v_6 - i_2 v_5 + i_3 v_6 . \end{aligned}$$

It is easily verified that the equations

$$\begin{aligned} L_1 \frac{di_1}{dt} &= E_1 - R_1 i_1 - v_4 + v_6 = \frac{\partial P}{\partial i_1}, \\ L_2 \frac{di_2}{dt} &= E_2 - R_2 i_2 - v_6 - v_5 = \frac{\partial P}{\partial i_2}, \\ L_3 \frac{di_3}{dt} &= -R_3 i_3 + v_6 = \frac{\partial P}{\partial i_3}, \\ C_1 \frac{dv_4}{dt} &= i_1 - f_1(v_4) = -\frac{\partial P}{\partial v_4}, \\ C_2 \frac{dv_5}{dt} &= i_2 - f_2(v_5) = -\frac{\partial P}{\partial v_5}, \\ C_3 \frac{dv_6}{dt} &= -i_1 + i_2 - i_3 = -\frac{\partial P}{\partial v_6} \end{aligned}$$

are the correct ones.

7. Limit situations. Although a complete circuit can, in some sense, be considered as typical, nevertheless it is sometimes more appropriate to neglect inductances or capacitances which are of minor importance. However, this may lead to circuits which are not complete. To see this we consider the complete circuit shown in Fig. 10. This

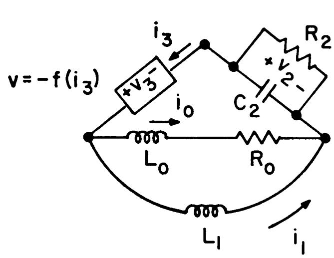

FIG. 10. A Complete Circuit When $L_0 \neq 0$

circuit is obviously complete since the currents of the lower branches determine the current $i_3 = (i_0 + i_1)$ in the upper branch. The mixed potential is easily computed to be

$$P(i_0, i_1, v_2) = -\tfrac{1}{2}R_0 i_0^2 - \int_0^{i_0+i_1} f(i)\, di + \frac{1}{2}\frac{v_2^2}{R_2} + v_2(i_0 + i_1), \tag{7.1}$$

which leads to the following differential equations.

$$L_0 \frac{di_0}{dt} = v_2 - f(i_0 + i_1) - R_0 i_0 = \frac{\partial P}{\partial i_0},$$

$$L_1 \frac{di_1}{dt} = -f(i_0 + i_1) + v_2 = \frac{\partial P}{\partial i_1},$$

$$C_2 \frac{dv_2}{dt} = -(i_0 + i_1) - \frac{v_2}{R_2} = -\frac{\partial P}{\partial v_2}.$$

However, if we let $L_0 \to 0$, then i_0 is no longer an independent variable and must be eliminated using $\partial P/\partial i_0 = 0$ or

$$v_2 = f(i_0 + i_1) + R_0 i_0 .$$

This leads to

$$v_2 + R_0 i_1 = f(i_0 + i_1) + R_0(i_0 + i_1),$$

or

$$i_0 + i_1 = h(v_2 + R_0 i_1),$$

and substituting this into the mixed potential, we obtain the new mixed potential in terms of the independent variables, i_1, v_2

$$Q(i_1, v_2) = -\tfrac{1}{2}R_0(h(v_2 + R_0 i_1) - i_1)^2 - \int_0^{h(v_2+R_0 i_1)} f(i)\, di + \frac{1}{2}\frac{v_2^2}{R_2} + v_2 h(v_2 + R_0 i_1). \tag{7.2}$$

This shows that the concept of a mixed potential is still meaningful,* but cannot be interpreted as easily as in the complete case.

It is obvious that one can add inductors in series and capacitors in parallel to a circuit to make it complete and then consider the original circuit as a limiting case of the new circuit so obtained. This procedure can be justified on physical grounds since these reactances are present anyway as parasitic elements.

Of course, the mixed potential of a noncomplete circuit is not necessarily so complicated as in our example. In determining stability criteria in the next section we will only require that the mixed potential have the form

$$P(i^*, v^*) = -A(i^*) + B(v^*) + (i^*, Dv^*), \tag{7.3}$$

where D is a constant matrix whose elements need not be ± 1, 0. Obviously this is the case for a complete circuit, and it is for this reason and the fact that all other circuits can be considered as limit cases of complete circuits that we have considered complete

*It is remarkable that the derivatives (e.g. with respect to v_2) of Q and P agree where $\partial P/\partial i_0 = 0$. This follows from the identity $\partial Q/\partial v_2 = \partial P/\partial v_2 + \partial P/\partial i_0 \; \partial i_0/\partial v_2$ and the observation that the last term vanishes for $\partial P/\partial i_0 = 0$. In other words instead of eliminating i_0 from the differential equations one can eliminate it from the potential function.

circuits in such detail. We emphasize, however, that completeness is certainly not necessary for the mixed potential to have the form of (7.3).

8. Nonoscillating circuits. We wish to consider circuits without time-varying elements and to establish criteria which rule out the existence of self-sustained oscillations. For a linear circuit with only positive resistors, no oscillations can occur if there is dissipation, but this is not true of nonlinear circuits as, for example, the van der Pol circuit. This question has practical significance in the design of computer circuits, control systems, etc., in which are utilized nonlinear elements and, in such cases, one wants to establish overall stability requirements. We shall see in this section how one can use the mixed potential to fulfill such requirements.

We have a system of differential equations which we write in vector form*

$$-Jx' = \partial P(x)/\partial x, \tag{8.1}$$

where

$$x = \begin{pmatrix} i \\ v \end{pmatrix},$$

$$J = \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix},$$

which is, in general**, an indefinite symmetric matrix, and $\partial P/\partial x$ denotes the gradient of P. We note that the stationary points of $P(x)$, i.e., where $\partial P(x)/\partial x = 0$, are exactly the equilibria of (8.1) and we want to discuss conditions under which *all* solutions approach these equilibria as $t \to \infty$. In particular, this would exclude the existence of periodic solutions if there are only a finite number of equilibria.

For this purpose, we differentiate $P(x)$ along the solutions of (8.1) and find

$$\frac{dP(x)}{dt} = (x', P_x) = -(x', Jx'). \tag{8.2}$$

If the circuit does not contain any inductors, then we have

$$\frac{dP}{dt} = -(x', Cx')$$

since $J = C$ and obviously P decreases along solutions except at equilibria. Following Liapounov's ideas, one could derive from this, assuming, in addition, that $P(x) \to \infty$ as $|x| \to \infty$, that every solution tends to one of the equilibrium points as $t \to \infty$. Similarly, if there are no capacitors in the circuit, then

$$\frac{d(-P)}{dt} = -(x', Lx'),$$

and assuming that $-P(x) \to \infty$ as $|x| \to \infty$, we have asymptotic stability again.

It is intuitively clear that if L or C is sufficiently small, then a similar result should hold also. Just how large L or C can become without causing oscillations will be answered in the following investigation.

We first ask whether one can describe the system (8.1) by another pair J^*, P^*† in

*Here x' denotes dx/dt.

**If no inductors are present, J is, of course, positive definite.

†The notation J^* should not be confused with the adjoint of the matrix J.

place of J, P such that

$$-J^*x' = P^*_x(x), \tag{8.3}$$

and such that J^* is positive definite. Since $-x' = J^{-1}P_x(x)$, a necessary and sufficient condition for a new pair J^*, P^* describing the differential equations in the form (8.3) is

$$J^*J^{-1}P_x = P^*_x\,. \tag{8.4}$$

Our aim is to find a pair J^*, P^* such that (x', J^*x') is positive definite and $P^*(x) \to \infty$ for $|x| \to \infty$.

We note first if (J_1, P_1), (J_2, P_2) are two pairs describing (8.1), then so are

$$(\alpha J_1 + \beta J_2, \alpha P_1 + \beta P_2), \tag{8.5}$$

which gives considerable freedom in constructing other pairs.

To find one nontrivial pair other than (J, P) we observe that if M is any constant *symmetric* matrix, then the pair

$$J^* = P_{xx}MJ, \qquad P^* = \tfrac{1}{2}(P_x, MP_x),$$

is a possible choice. This is easily seen since

$$P^*_x = (P_{xx}M)P_x$$

and therefore

$$J^*J^{-1}P_x = P_{xx}MJJ^{-1}P_x = P^*_x$$

which by (8.4) implies (8.3). By superposition we obtain the more general pairs

$$J^* = (\lambda I + P_{xx}M)J, \qquad P^* = \lambda P + \tfrac{1}{2}(P_x, MP_x) \tag{8.6}$$

where M ranges over all constant symmetric matrices and λ is an arbitrary constant.

Having made these observations, we shall now prove some theorems which depend on the mixed potential having the form

$$P(i, v) = -A(i) + B(v) + (i, \gamma v), \tag{8.7}$$

where γ is a constant matrix *not* necessarily with elements ± 1, 0.

We first consider a "semilinear" case, i.e.,

$$P(i, v) = -\tfrac{1}{2}(i, Ai) + B(v) + (i, \gamma v - a), \tag{8.8}$$

where A is a constant symmetric matrix and a is a constant vector so that the first r equations in (4.10) are linear. In case the circuit is complete, this means that all the nonlinear resistors are in the subnetwork $\mathfrak{N}_v$. In theorems 3 and 4 which follow, we can allow nonlinear inductors, i.e.,

$$L = L(i),$$

and nonlinear capacitors, i.e.,

$$C = C(v).$$

However, it is necessary to assume that $L(i)$ and $C(v)$ are symmetric, positive definite, and their least eigenvalues are bounded away from zero.

Theorem 3. If A is positive definite, $B(v) + |\gamma v| \to \infty$ as $|v| \to \infty$, and

$$\| L^{1/2}(i)A^{-1}\gamma C^{-1/2}(v) \| \leq 1 - \delta, \qquad \delta > 0,^* \tag{8.9}$$

*The notation $\| K \|^2$ of a matrix K denotes $\max_{|x|=1} (Kx, Kx)$.

for all i, v, then all solutions of (8.1) tend to the set of equilibrium points as $t \to \infty$.

Proof. We choose M and λ in (8.6) as follows:

$$M = \begin{bmatrix} 2A^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \lambda = 1.$$

Then

$$J^* = J + P_{xx}MJ = \begin{bmatrix} L & 0 \\ -2\gamma^T A^{-1}L & C \end{bmatrix},^*$$

and

$$(x', J^*x') = (y, y) - 2(z, C^{-1/2}\gamma^T A^{-1}L^{1/2}y) + (z, z),$$

where

$$y = L^{1/2}\frac{di}{dt} \quad \text{and} \quad z = C^{1/2}\frac{dv}{dt}.$$

With $K = L^{1/2}A^{-1}\gamma C^{-1/2}$, we have

$$(x', J^*x') = (y - Kz, y - Kz) + (z, z) - (Kz, Kz),$$

and since $\| K \| \leq 1 - \delta$, then

$$(x', J^*x') \geq | y - Kz |^2 + \delta | z |^2 \geq 0,$$

which is zero if and only if $di/dt = dv/dt = 0$. Thus

$$P^* = P + (P_i , A^{-1}P_i)$$

is monotone decreasing except at equilibria and it remains to be proved that $P^*(x) \to \infty$ as $| x | \to \infty$.

To show this we rewrite P in the form

$$P(\alpha, v) = -\tfrac{1}{2}(\alpha, A^{-1}\alpha) + U(v),$$

where $\alpha = P_i = -Ai + \gamma v - a$ and

$$U(v) = \tfrac{1}{2}[(a - \gamma v), A^{-1}(a - \gamma v)] + B(v). \tag{8.10}$$

Then P^* becomes

$$P^*(\alpha, v) = \tfrac{1}{2}(\alpha, A^{-1}\alpha) + U(v).$$

Since A is positive definite and $B(v) + | \gamma v | \to \infty$ as $| v | \to \infty$, it is clear also that $U(v) \to \infty$ as $| v | \to \infty$. It remains to be shown that $| \alpha | + | v | \to \infty$ as $| i | + | v | \to \infty$. For this purpose we consider the matrix

$$S = \begin{bmatrix} -A & \gamma \\ 0 & I \end{bmatrix},$$

which is nonsingular and gives rise to the transformation

$$\begin{bmatrix} \alpha + a \\ \omega \end{bmatrix} = \begin{bmatrix} -A & \gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix},$$

*The notation γ^T denotes the transpose of the matrix γ.

or

$$y = Sx.$$

Hence $(y, y) = (Sx, Sx) \geq 0$. However, S is nonsingular and its least eigenvalue is bounded away from zero so that $(Sx, Sx) > 0$ for $x \neq 0$. Since $(Sx, Sx) \geq \lambda_1(x, x)$ where λ_1 is the smallest eigenvalue of $S^T S$, then we have

$$(y, y) = (Sx, Sx) \geq \lambda_1(x, x),$$

where $\lambda_1 > 0$. This gives

$$|\alpha + a|^2 + |\omega|^2 \geq \lambda_1(|i|^2 + |v|^2),$$

which implies $|\alpha| + |v| \to \infty$ for $|i| + |v| \to \infty$. By applying a well-known theorem of Liapounov (see LaSalle and Lefschetz [12]), we conclude that every solution of (8.1) approaches the set of equilibrium points as $t \to \infty$.

Next we consider the other semilinear case, i.e.,

$$P(i, v) = -A(i) + \tfrac{1}{2}(v, Bv) + (v, \gamma^T i + b), \tag{8.11}$$

where B is a constant symmetric matrix and b is a constant vector so that the last s equations of (4.10) are linear. In case the circuit is complete, this means that all the nonlinear resistors are in the subnetwork $\mathfrak{N}_i$.

Theorem 4. If B is positive definite, $A(i) + |\gamma^T i| \to \infty$ as $|i| \to \infty$, and

$$\| C^{1/2}(v) B^{-1} \gamma^T L^{-1/2}(i) \| \leq 1 - \delta, \qquad \delta > 0, \tag{8.12}$$

for all i, v, then all solutions of (8.1) tend to the set of equilibrium points as $t \to \infty$.

Proof. Since the proof is similar to the proof of theorem 3, we will only indicate it. We choose M and λ in (8.6) as follows:

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 2B^{-1} \end{pmatrix}, \qquad \lambda = -1.$$

Then it is easy to show that $P^* = -P + (P_v, B^{-1} P_v)$ is monotone decreasing along solutions of (8.1). To show that $P^*(i, v) \to \infty$ as $|i| + |v| \to \infty$, we write P^* in the form

$$P^*(i, v) = \tfrac{1}{2}(P_v, B^{-1} P_v) + W(i),$$

where

$$W(i) = \tfrac{1}{2}[(b + \gamma^T i), B^{-1}(b + \gamma^T i)] + A(i), \tag{8.13}$$

and proceed as in the proof of theorem 3.

We remark that it is not obvious, in general, when the first two conditions of theorems 3 and 4 hold. In section 19, part II, conditions on the network and the resistors are given which are equivalent to these conditions if the circuit is complete.

The next theorem does not require semilinearity but does require that the matrices L and C are constant symmetric and positive definite. The mixed potential has the form (8.7) where nonlinearities may occur in both $A(i)$ and $B(v)$. For this theorem we construct the function

$$P^*(i, v) = \left(\frac{\mu_1 - \mu_2}{2}\right) P(i, v) + \tfrac{1}{2}(P_i, L^{-1} P_i) + \tfrac{1}{2}(P_v, C^{-1} P_v), \tag{8.14}$$

where μ_1 is the smallest eigenvalue of the matrix $L^{-1/2}A_{ii}(i)L^{-1/2}$ for all i and μ_2 is the smallest eigenvalue of $C^{-1/2}B_{vv}(v)C^{-1/2}$ for all v. We shall use $\mu(M)$ to denote the smallest eigenvalue of a symmetric matrix M.

Theorem 5. If

$$\mu(L^{-1/2}A_{ii}(i)L^{-1/2}) + \mu(C^{-1/2}B_{vv}(v)C^{-1/2}) \geq \delta, \qquad \delta > 0, \tag{8.15}$$

for all i, v and

$$P^*(i, v) \to \infty \quad \text{as} \quad |\, i \,| + |\, v \,| \to \infty, \tag{8.16}$$

where $P^*(i, v)$ is given by (8.14), then all solutions of (8.1) approach the equilibrium solutions as $t \to \infty$.

It is difficult, in general, to replace (8.16) by simple conditions on $P(i, v)$. We leave it therefore in the form given since it can be checked directly.

Proof. For M and λ in (8.6) we choose

$$M = \begin{bmatrix} L^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \quad \text{and} \quad \lambda = \frac{\mu_1 - \mu_2}{2}.$$

Then

$$J^* = \begin{bmatrix} A_{ii} & \gamma \\ -\gamma^T & B_{vv} \end{bmatrix} + \lambda \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}.$$

With $z = L^{1/2}\,(di/dt)$ and $w = C^{1/2}\,(dv/dt)$ we have

$$\begin{aligned} (x', J^*x') &= (z, L^{-1/2}A_{ii}L^{-1/2}z) + (w, C^{-1/2}B_{vv}C^{-1/2}w) + \lambda((w, w) - (z, z)) \\ &\geq (\mu_1 - \lambda)(z, z) + (\mu_2 + \lambda)(w, w) \\ &\geq \tfrac{1}{2}(\mu_1 + \mu_2)[(z, z) + (w, w)] \geq 0. \end{aligned}$$

Since $\mu_1 + \mu_2 \geq \delta > 0$ by assumption, then (x', J^*x') is positive definite and equal to zero if and only if $di/dt = dv/dt = 0$. Thus, $P^*(i, v)$, given by (8.14), is monotone decreasing except at the equilibria.

In summary, we have three theorems which give sufficient conditions for asymptotic stability in the large. Theorems 3 and 4 give conditions which depend on the graph of the network as given by the matrix γ but are independent of the nonlinearities. On the other hand, theorem 5 gives conditions which depend on the nonlinearities but are independent of the graph γ of the network.

9. Example. In this section we consider a large ladder network shown in the following figure, and we wish to apply theorem 3 to find conditions for nonoscillation. Then, in order to demonstrate that these criteria are sharp, we will choose a particular nonlinear element for this circuit and exhibit an exact periodic solution.

We will see that the conditions which cause this oscillation can be chosen very close to the nonoscillatory conditions and also, in one case, that the point separating oscillation and nonoscillation approaches zero as the number of loops in the circuit becomes large.

The circuit considered is shown in Fig. 11. The nonlinear element is given by $g(v)$ which is the current through the element in the direction shown.

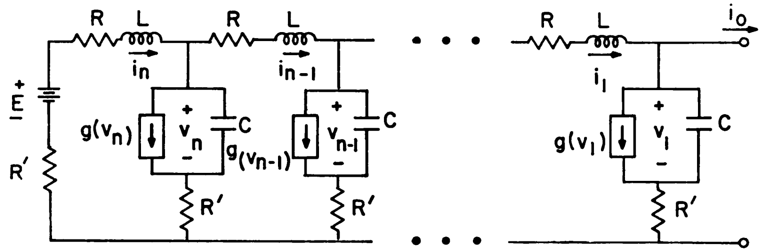

FIG. 11. Large Nonlinear Ladder Circuit

Assuming that $v_{n+1} = E$ and $i_0 = 0$, the mixed potential for the circuit is

$$P(i, v) = -\tfrac{1}{2}R' i_n^2 + \sum_{k=1}^{n} \left[-\tfrac{1}{2}R i_k^2 - \tfrac{1}{2}R'(i_{k-1} - i_k)^2 + i_k(v_{k+1} - v_k) + \int_0^{v_k} g(v)\, dv \right],$$

or written in vector notation

$$P(i, v) = -\tfrac{1}{2}R(i, i) - \tfrac{1}{2}R'(\alpha i, \alpha i) + (i, \gamma v) + B(v) - (a, i),$$

where α is an $n + 1$ by n matrix and γ is an n by n matrix given by

$$\alpha = \begin{bmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & & & \vdots \\ 0 & 1 & -1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ \vdots & & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & -1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ -E \end{bmatrix},$$

and

$$B(v) = \sum_{k=1}^{n} \int_0^{v_k} g(v)\, dv.$$

In applying theorem 3 we shall consider two cases separately: $R' = 0$ and $R = 0$. With $R' = 0$, we want to show first that the circuit has only a finite number of equilibrium solutions. The equilibrium equations are

$$\frac{v_{k-1} - 2v_k + v_{k+1}}{R} = g(v_k), \qquad k = 2, \cdots, n,$$

$$\frac{v_2 - v_1}{R} = g(v_1),$$

where $v_{n+1} = E$. The solution is found graphically as shown in Fig. 12, where, for example, we have chosen $g(v)$ to be the characteristic for a tunnel diode. We see that there are at most three intersections and hence the number of equilibria is at most 3^n.

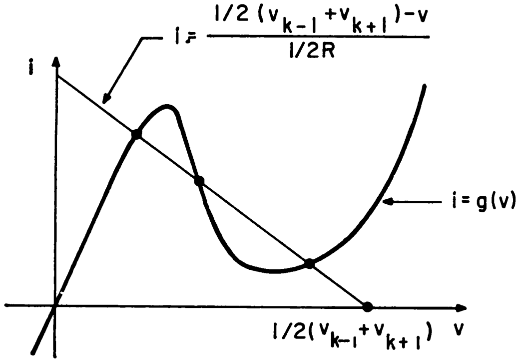

FIG. 12. Equilibrium Solutions for the k^{th} Circuit

In checking the conditions of theorem 3, it is clear that $B(v) \to \infty$ as $|v| \to \infty$ and since $A = RI$,* then A is positive definite. We now compute the norm of the matrix $K = L^{1/2}A^{-1}\gamma C^{-1/2}$ or in this case $K = (L^{1/2}/RC^{1/2})\gamma$. Computing the norm of γ, we find $\|\gamma\|^2 \le 4$ and therefore

$$\|K\|^2 \le \frac{4L}{R^2C}.$$

Hence, from theorem 3, $\|K\|^2 < 1$ for nonoscillation is satisfied if

$$\frac{L}{R^2C} < \frac{1}{4}. \tag{9.1}$$

For the case $R = 0$, the equilibrium equations are of the form

$$\frac{E - R' \sum_{l \ne k} i_l - v_k}{2R'} = g(v_k), \qquad k = 1, \cdots, n,$$

and it is clear again that there are at most 3^n equilibrium solutions. In this case the matrix A is

$$A = R'\alpha^T\alpha = R' \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix},$$

which is positive definite since α has full rank. One computes

* I denotes the identity matrix.

$$R'(-\gamma^{-1}A)^{-1} = -R'A^{-1}\gamma$$

$$= \frac{1}{(n+1)} \left[\begin{array}{ccccccc} n & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ n-1 & n-1 & -2 & \cdots & \cdots & \cdots & -2 \\ n-2 & n-2 & n-2 & -3 & \cdots & \cdots & -3 \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ 3 & \cdots & \cdots & \cdots & 3 & 2-n & 2-n \\ 2 & \cdots & \cdots & \cdots & \cdots & 2 & 1-n \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{array}\right].$$

In order to estimate the norm of $K = L^{1/2}A^{-1}\gamma C^{-1/2}$, it is necessary to estimate the norm of $-R'A^{-1}\gamma$. We compute

$$\begin{aligned}(-R'A^{-1}\gamma x, -R'A^{-1}\gamma x) &= \frac{1}{(n+1)^2}\sum_{l=1}^{n}\left[-l\sum_{k=l+1}^{n} x_k + (n-l+1)\sum_{k=1}^{l} x_k\right]^2 \\ &\leq \frac{n}{(n+1)^2}\sum_{l=1}^{n}\left[\sum_{k=1}^{l}(n-l+1)^2x_k^2 + \sum_{k=l+1}^{n} l^2x_k^2\right] \\ &\leq \frac{n}{(n+1)^2}\sum_{l=1}^{n}[(n-l+1)^2+l^2]\sum_{k=1}^{n} x_k^2 \\ &\leq \frac{n}{(n+1)^2}\left(\sum_{k=1}^{n} x_k\right)\sum_{l=1}^{n}(n+1)^2 \\ &\leq n^2(x, x).\end{aligned}$$

Thus $\| -R'A^{-1}\gamma \|^2 \leq n^2$.

According to theorem 3, we are forced to require

$$\frac{L}{R'^2C} < \frac{1}{n^2} \tag{9.2}$$

to ensure $\| K \|^2 < 1$, which is sufficient for nonoscillation. In contrast to the previous criterion for $R' = 0$, this condition is more and more stringent as n increases. That this is not only due to our estimates will be seen from the following consideration.

We now consider a particular nonlinearity for $g(v)$ which is the piecewise linear function shown in Fig. 13. Our purpose is to find a periodic solution such that the magnitude

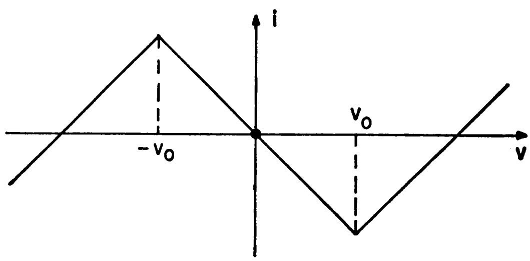

FIG. 13. A Special Choice for $g(v)$

of the voltage across any diode is less than v_0 . Then we can replace $g(v)$ by $-Gv$ for $|v| < v_0$ where G is a positive constant, and we have the following differential equations to solve:

$$L\frac{di_k}{dt} = -Ri_k + R'(i_{k+1} - 2i_k + i_{k+1}) - (v_k - v_{k+1}),$$

$$C\frac{dv_k}{dt} = -(i_{k-1} - i_k) + Gv_k, \qquad k = 1, \cdots, n, \tag{9.3}$$

with the boundary conditions $v_{n+1} = E = 0$, $i_0 = i_{n+1} = 0$.

Assuming the solution is of the form

$$i_k = ae^{\alpha t}\sin k\Delta,$$

$$v_k = be^{\alpha t}\cos(k - \tfrac{1}{2})\Delta, \tag{9.4}$$

we find from the boundary conditions that

$$\cos(n + \tfrac{1}{2})\Delta = 0$$

or

$$\Delta = \Delta_\nu = \left(\frac{\nu + \frac{1}{2}}{n + \frac{1}{2}}\right)\pi, \qquad \nu = 0, \cdots, n - 1. \tag{9.5}$$

Note that $\nu = n$ is excluded because this corresponds to the identically zero solution. We denote $2\sin\Delta_\nu/2$ by λ_ν and substituting (9.4) into (9.3) we find the following simultaneous equations for a and b:

$$\begin{aligned} La\alpha &= -Ra - \lambda_\nu^2 R'a + b\lambda_\nu, \\ Cb\alpha &= -\lambda_\nu a + Gb. \end{aligned} \tag{9.6}$$

This has a nontrivial solution if and only if the determinant of the coefficients of a and b is zero, i.e.,

$$\alpha^2 + \left[\frac{R}{L} + \frac{R'}{L}\lambda_\nu^2 - \frac{G}{C}\right]\alpha + \frac{1}{LC}[\lambda_\nu^2 - G(R + R'\lambda_\nu^2)] = 0.$$

Oscillations occur if α is purely imaginary and nonzero, which means

$$\frac{R}{L} + \frac{R'}{L}\lambda_\nu^2 = \frac{G}{C}, \tag{9.7}$$

and

$$\lambda_\nu^2 > G(R + R'\lambda_\nu^2). \tag{9.8}$$

Eliminating G, these conditions become

$$\frac{R}{\lambda_\nu} + R'\lambda_\nu \le \sqrt{\frac{L}{C}}. \tag{9.9}$$

The left-hand side is plotted as a function of λ in Fig. 14.

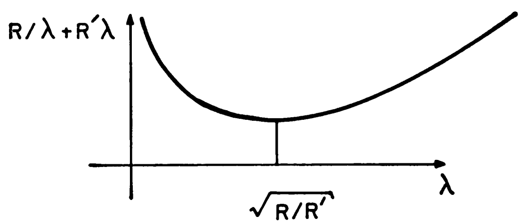

FIG. 14. $R/\lambda + R'\lambda$ versus λ

We also see that the condition $| v | \leq | v_0 |$ is easily satisfied; for purely imaginary α, this becomes $| b | \leq v_0$, and we are at liberty to choose b since equation (9.6) is homogeneous.

For $R' = 0$ condition (9.9) becomes

$$\frac{L}{R^2 C} > \frac{1}{\lambda_\nu^2}, \tag{9.10}$$

and, for the purposes of this example, we want to choose ν such that λ_ν is maximum. This leads to the choice $\nu = n - 1$ so that

$$\lambda_{n-1}^2 = 4 \sin^2 \frac{n - 0.5}{n + 0.5} \frac{\pi}{2} \approx 4 - \frac{4\pi^2}{(2n + 1)^2} \quad \text{for large } n,$$

and (9.10) becomes

$$\frac{L}{R^2 C} > \frac{1}{4}\left[1 - \frac{\pi^2}{(2n + 1)^2}\right]^{-1} \quad \text{for large } n. \tag{9.11}$$

Thus, under this condition, an oscillatory solution exists and is given by (9.4).

In the case $R = 0$ equation (9.9) becomes

$$\frac{L}{R'^2 C} > \lambda_\nu^2, \tag{9.12}$$

and we make λ_ν^2 a minimum by choosing $\nu = 0$. Then

$$\lambda_\nu^2 = 4 \sin^2 \frac{1}{(2n + 1)}\left(\frac{\pi}{2}\right) \approx \frac{\pi^2}{(2n + 1)^2} \quad \text{for large } n,$$

and condition (9.12) becomes

$$\frac{L}{R'^2 C} > \frac{\pi^2}{(2n + 1)^2} \quad \text{for } n \text{ large}. \tag{9.13}$$

Thus, under this condition, an oscillatory solution exists and is given by (9.4).

In summary, we have shown in the case $R' = 0$ that under the condition

$$\frac{L}{R^2 C} < \frac{1}{4},$$

no oscillations can exist, but if

$$\frac{L}{R^2 C} > \frac{1}{4}\left[1 - \frac{\pi^2}{(2n + 1)^2}\right]^{-1} \quad \text{for large } n,$$

and if we choose $g(v)$ appropriately, then a periodic solution actually does exist. This shows that the criterion of theorem 3 is fairly sharp, and, in fact, it is the best possible in general.

In the case $R = 0$, no oscillations can occur if

$$\frac{L}{R'^2 C} < \frac{1}{n^2},$$

but if

$$\frac{L}{R'^2 C} > \frac{\pi^2}{(2n+1)^2} \quad \text{for large } n,$$

and if $g(v)$ is chosen appropriately, then a periodic solution exists. We also see in this case that the tendency for this circuit to oscillate increases as the number of loops increases.

This discussion suggests the investigation of the continuous analog of (9.3) which corresponds to a nonlinear transmission line. If x in $0 \leq x \leq 1$ is a variable along the line, the equations take the form

$$L \frac{\partial i}{\partial t} = -Ri + R' \frac{\partial^2 i}{\partial x^2} + \frac{\partial v}{\partial x},$$

$$C \frac{\partial v}{\partial t} = \frac{\partial i}{\partial x} + g(v),$$

with the boundary conditions

$$i(t, 0) = 0, \qquad v(t, 1) = E.$$

Here R, R', L, C represent some appropriate densities of resistance, inductance, and capacitance. For $R = R' = 0 = g(v)$, one obtains the equations for a lossless transmission line. As in the discrete case, one can attempt to describe the solutions for large values of t. For instance, the function P would become an integral

$$\int_0^1 \left[-\frac{R}{2} i^2 + R' i_x^2 + i v_x + G(v(t, x)) \right] dx, \qquad G(v) = \int_0^v g(v)\, dv.$$

However, the detailed discussion is out of place here. We just mention that for $R = 0$ and appropriate $g(v)$, E, there is a continuum of equilibrium solutions, namely any function $v(x)$, for which

$$-\frac{v}{R'} + g(v) = \text{constant} = -\frac{E}{R'} + g(E)$$

holds, gives rise to an equilibrium solution. If $-v/R' + g(v) = $ constant has several roots, v can be a step function taking on these roots in arbitrary intervals.

10. Solutions near an equilibrium. The question of the stability of an equilibrium solution can be studied by two methods. One is the standard method of investigating the structure of the variational equations, and the other is a method which uses the functions P^* of section 8. In this section we discuss both of these methods separately.

A. We study the solutions of the system

$$L \frac{di}{dt} = \frac{\partial P}{\partial i}, \qquad C \frac{dv}{dt} = -\frac{\partial P}{\partial v} \tag{10.1}$$

near an equilibrium $i = i_0$, $v = v_0$. For this purpose, we replace the above equations by the linearized system (variational equations). This system will be simplified by introducing

$$x = [L(i_0)]^{1/2}(i - i_0) \qquad y = [C(v_0)]^{1/2}(v - v_0).$$

Then

$$\frac{dx}{dt} = Ax - By, \qquad \frac{dy}{dt} = B^T y + Dy, \tag{10.2}$$

where

$$A = L^{-1/2}P_{ii}L^{-1/2}, \qquad B = -L^{-1/2}P_{iv}C^{-1/2}, \qquad D = -C^{-1/2}P_{vv}C^{-1/2}$$

at $i = i_0$, $v = v_0$. Here, A and D are symmetric matrices.

The properties of the matrix

$$M = \begin{bmatrix} A & -B \\ B^T & D \end{bmatrix} \tag{10.3}$$

are most easily discussed in terms of the indefinite bilinear form

$$-(x, x') + (y, y'). \tag{10.4}$$

Combining x, y into one vector

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad z' = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

(10.4) can be written as

$$(z, Jz') = -(x, x') + (y, y'), \tag{10.5}$$

where

$$J = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

The matrix M of (10.3) is then symmetric with respect to the form (z, Jz'), i.e.,

$$\begin{aligned} (Mz, Jz') &= (Ax - By, -x') + (Bx + Dy, y') \\ &= -(Ax, x') + (By, x') + (Bx, y') + (Dy, y') \\ &= (Mz', Jz). \end{aligned}$$

It is well-known that the behavior of the solutions of the system (10.2) is described by the eigenvalues of the matrix M. For instance, the stability of the equilibrium is ensured if all the eigenvalues of M have negative real parts. We wish to study which restrictions the symmetry properties of M impose on its eigenvalues. A matrix which is symmetric with respect to a definite form has only real eigenvalues. Since the form (10.5) is indefinite, such a statement does not hold for M. However, we will prove the following lemma.

Lemma. If z is a (complex) eigenvector of M corresponding to a nonreal eigenvalue λ, then

$$(z, J\bar{z}) = -|x|^2 + |y|^2 = 0.$$

Proof. From $Mz = \lambda z$ follows

$$(Mz, J\bar{z}) = \lambda(z, J\bar{z}) = \lambda(|y|^2 - |x|^2),$$

and since M is symmetric with respect to (10.5), this expression must be real. Hence,

$$(\operatorname{Im} \lambda)(-|x|^2 + |y|^2) = 0,$$

which proves the lemma.

For such matrices as (10.3) the condition of section 8, theorem 3,

$$\| K \| < 1,$$

takes the form

$$\| A^{-1}B \| < 1. \tag{10.6}$$

We want to investigate the restrictions on the eigenvalues of M imposed by condition (10.6).

Theorem 6. If in the matrix M of (10.3) the matrices A, D are symmetric, $-A$ is positive definite and (10.6) holds, then the eigenvalues of M lie either in the left-half plane or on the real line.

Remark. Note that D can be negative definite since the only restriction on D is symmetry. Therefore, one cannot expect that the equilibrium is stable, in general. In fact, in case the nonlinear differential equations have several equilibria, some of them must be unstable and give rise to matrices M with positive eigenvalues.

Since the eigenvalues are not purely imaginary, periodic solutions are excluded. Moreover, the solutions escaping from the unstable equilibria are not oscillatory.

Proof. If λ is not real and if

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \neq 0,$$

is the corresponding complex eigenvector, then according to the lemma

$$|x| = |y|,$$

and from $Mz = \lambda z$ follows

$$(A - \lambda I)x = By,$$

or

$$(I - \lambda A^{-1})x = A^{-1}By.$$

If we normalize $| x | = | y | = 1$, we have

$$\| (I - \lambda A^{-1})x \|^2 = \theta < 1,$$

where $\theta = \| A^{-1}B \|^2$. This gives

$$| x |^2 - 2(\mathrm{Re}\,\lambda)(\bar{x}, A^{-1}x) + | \lambda |^2 \, | A^{-1}x |^2 = \theta,$$

or

$$| \lambda |^2 \, | A^{-1}x |^2 - 2(\mathrm{Re}\,\lambda)(\bar{x}, A^{-1}x) = \theta - 1 < 0. \tag{10.7}$$

Dropping the first term, one finds

$$(\mathrm{Re}\,\lambda) < 0,$$

since $(\bar{x}, A^{-1}x)$ is negative by assumption.

More precisely, for each eigenvector of M, (10.7) defines a circle in the left-half λ-plane, and the corresponding eigenvalue must lie on this circle or on the real axis. (See Fig. 15).

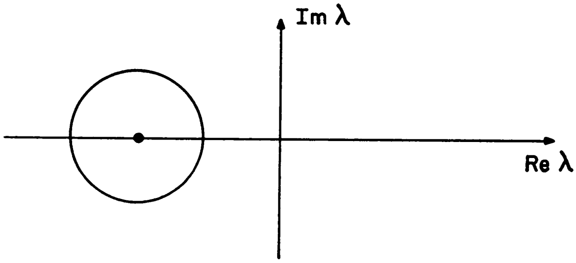

FIG. 15. Circle Containing Eigenvalues of M

The condition of theorem 4 leads to a similar restriction on the eigenvalues.

Theorem 7. If $\mu(A)$ and $\mu(D)$ denote the maximal eigenvalues of the symmetric matrices A, D and if

$$\mu(A) + \mu(D) < 0,$$

then all nonreal eigenvalues of M lie in $\mathrm{Re}\,\lambda < 0$.

Proof. With the above notation we have

$$Ax - By = \lambda x,$$

$$B^T\bar{x} + D\bar{y} = \bar{\lambda}\bar{y},$$

if λ is a nonreal eigenvalue. Multiplying the first equation by $\bar{x}$ and the second by y and adding, we find

$$(\bar{x}, Ax) + (y, D\bar{y}) = \lambda \, | x |^2 + \bar{\lambda} \, | y |^2.$$

From the lemma $| x | = | y |$ and normalizing $| x | = | y | = 1$, we have

$$(\bar{x}, Ax) + (y, D\bar{y}) = 2 \, \mathrm{Re}\,\lambda.$$

By assumption, the left-hand side remains negative, namely,

$$0 > \mu(A) + \mu(D) \geq 2 \operatorname{Re} \lambda,$$

which proves the theorem.

The linearized equations also show that the conditions of theorems 3, 4, and 5 are sharp. For instance, suppose that x and y in equations (10.2) are scalar. Then M has two eigenvalues which under the condition

$$\text{trace} \quad M = A + D = 0,$$

are both imaginary, and hence the equations have only oscillatory solutions. The condition of theorem 5 for nonoscillation is simply

$$A + D < 0.$$

B. The stability of an equilibrium can also be decided by studying one of the functions P^* of section 8. It is easily verified that in all cases the stationary points of P^* coincide with the equilibrium solutions.

From a well-known theorem due to Dirichlet and Lagrange (and later exploited by Liapounov), if a function exists which decreases with time along the solutions except the equilibrium solutions, then an equilibrium solution is stable if and only if the function has a local minimum there. For a precise formulation of this statement and its proof we refer to the book of Chetayev [13]. Thus, we have

Theorem 8. Under the conditions of theorem 3, 4, or 5, the equilibria coincide with the stationary points of P^* and the local minima of P^* coincide with the stable equilibrium solutions.

In the semilinear case we can discuss the stability of the equilibrium solutions in terms of a function of the voltages only or of the currents only. For instance, in theorem 3 we had by (8.10)

$$P^*(i, v) = \tfrac{1}{2}(P_i \, , A^{-1}P_i) + U(v),$$

and it is obvious that the stationary points are given by

$$P_i = 0 \quad \text{and} \quad U_v = 0.$$

Since the first relation is linear and hence trivial, the equilibrium solutions can be obtained as stationary points of $U(v)$. Moreover, since A is positive definite, the local minima of $U(v)$ are the stable equilibria.

The advantage of this approach is that frequently it is much easier to determine the minima of a function than it is to determine the eigenvalues of the linearized equations.

Bibliography

[1] W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand Co., Inc., Princeton, N. J., 1945

[2] E. A. Guillemin, *Introductory circuit theory*, J. Wiley and Sons, Inc., N. Y., 1958

[3] E. J. Cartan, *Leçons sur les invariants integraux*, A. Hermann et Fils, Paris, 1922

[4] R. Duffin, *Nonlinear networks III*, Bull. Amer. Math. Soc. **54**, (1948) 119

[5] E. Goto et al., *Esaki diode high-speed logical circuits*, IRE Trans. on Electronic Computers *EC-9* (1960) 25

[6] J. Moser, *Bistable systems of differential equations*, Proc. of the Rome Symposium, Provisional International Computation Centre, Birkhäuser Verlag, 1960, pp. 320–329

[7] J. Moser, *Bistable systems with applications to tunnel diodes*, IBM J. Res. Dev. **5** (1961) 226
[8] B. D. H. Tellegen, *A general network theorem with applications*, Phillips Research Reports **7** (1952) 259
[9] L. Esaki, *New phenomenon in narrow Ge p-n junctions*, Phys. Rev. **109** (1958) 603
[10] W. Millar, *Some general theorems for nonlinear systems possessing resistance*, Phil. Mag. **42** (1951) 1150
[11] C. Cherry, *Some general theorems for nonlinear systems possessing reactance*, Phil. Mag. **42** (1951) 1161
[12] J. LaSalle and S. Lefschetz, *Stability by Liapunov's direct method with applications*, Academic Press, New York, London, 1961, p. 66
[13] N. G. Chetayev, *Stability of motion*, Moscow, 1946