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CHARACTERIZATION AND CALCULATION OF STEADY, COMPRESSIBLE, DIABATIC FLOW FIELDS*

BY

DONALD R. CHENOWETH

Lockhead Missiles and Space Co., Huntsville Research and Engineering Center

Introduction. Several physical and chemical phenomena invalidate the assumption of adiabatic flow in many compressible flow problems. When such processes are included in the mathematical model, the difficulties of calculation usually mask the important non-adiabatic nature of the flow. Hence, the effects of the variation of total temperature along the streamlines is often studied without direct reference to the mechanisms responsible for this behavior. These inviscid, non-conducting, steady flows with energy addition by heat sources are termed "diabatic" and correspond to heating processes which are thermodynamically reversible. The results from diabatic flow studies provide the basic insight into heat addition effects which is necessary before investigating more complicated problems where such phenomena as viscosity, conduction, diffusion, changes in gas composition, and electromagnetic effects are considered.

Some fluid dynamics problems related to combustion processes and involving changes in total temperature were first formulated correctly in the independent work of Chapman [1] and Jouguet [2] at the early part of this century. In a later paper, with application to meteorology, Kiebel [3] gave a complete classification of viscous, compressible flow with energy addition into thirteen dynamically permissible categories. In spite of the early recognition of the importance of heating effects, the discipline of diabatic flow has essentially been developed since 1944. Only a few representative papers and those which are necessary to place this work in proper perspective with respect to past investigations will be discussed here, but a more complete summary of the existing literature on diabatic flow can be found elsewhere [4].

Compressible flow textbooks usually present only the fundamentals of one-dimensional "simple-heating", although Tsien [5] and Krzywoblocki [6] present discussions of some of the more general aspects of diabatic flow. Most of the basic ideas about the effects of heat addition on the flow properties, the behavior of the streamlines at the sonic condition, and the phenomenon of thermal choking were obtained from the early one-dimensional studies [7, 8, 9, 10]. It was established [11, 12, 13, 14] that the effects of localized heat sources are much like those of fluid sources. Such ideas led to the patent

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application [13] which suggested many applications of heating in the neighborhood of airfoils and other aerodynamic bodies. Since then several investigations have been conducted on the problem of heat addition under airfoils using graphical [15, 16], analytical [17, 18] and experimental [19] techniques. Agreement between the results from these studies has not been particularly good. The theory was usually linearized for very small heat addition rates or one-dimensional flow approximations were introduced. The so-called direct approach to diabatic flow calculations is characterized by such linearizing approximations. For instance, in some recently published work [20], Marsh and Horlock investigated diabatic flow in which temperature gradients were assumed small (first-order), such that non-uniform heat addition across the stream produced only second-order (negligible) changes in vorticity.

Indirect approaches, where the heating function is not specified, but calculated from prior knowledge of the flow pattern have also been used. For example, Starr [21] studied two-dimensional motion of the atmosphere and integrated the equations exactly for circular vortex motion involving the presence of heat sources and sinks. The flow was assumed to be compressible and a governing equation was expressed in terms of the stream function and the specific volume. The stream function was assumed given so that a linear, second-order, hyperbolic type partial differential equation with specific volume as dependent variable was obtained.

An inverse method which differed from that of Starr's was presented by Hicks [22] where the introduction of a potential function allowed the retention of some control over the heat sources. This inverse method consisted of calculating the flow properties from prior knowledge of a flow pattern satisfying a governing potential equation. To investigate specific questions arising as a result of an earlier paper [23] where the governing equations were expressed alternately in terms of velocity, Mach, and Crocco vectors, Hicks [22] utilized an initially arbitrary vector **N**, parallel to the velocity vector. A wide variety of diabatic irrotational **N** flows (rotational velocity vector) was suggested in discussions of the choice of two arbitrary functions which were defined to give the "form" and "character" of the irrotational flows. It was demonstrated that the governing potential equation for these diabatic flows could remain of fixed type (elliptic for example) for all values of the local Mach number. Later [14], a linearization transformation due to Dimsdale was presented. The linearization transformation was completed [24, 25], and the results were used to calculate exactly some simple diabatic flow fields not previously considered. It was pointed out that these flows formed the basis for future generalization using the principle of superposition, thus allowing exact calculation of more complex diabatic flows.

In this paper, Hicks' inverse method is developed further both analytically and numerically. This generalization and extention is accomplished through a new diabatic flow formulation, the **L** theory, which is more simple than Hicks' **N** theory; however, the most interesting classes of flow are still contained in the simplified equations and a much deeper understanding is gained about these classes of diabatic flow. The mathematical advantage of this new specification over a similar approach utilizing either the velocity, Mach, Crocco, or N vector language is emphasized. Dimsdale's transformation, which was used in the **N** theory for special integrability functions, is carried out for an arbitrary integrability function through the use of the **L** formulation. It is shown that linear potential equations can be arrived at in two ways for special choices of the integrability function which appears in the equation of motion due to the **L** irrotationality condition. One corresponds to assuming that the so-called "kinetic flux vector", $\rho^{1/2}\mathbf{V}$, is irrotational; the other results in a generalized hodograph plane with the Legendre transform of the original potential function as the dependent variable. The flows governed by the latter potential equation are shown to be related to irrotational velocity vector flows by a similarity law, and the flows with irrotational kinetic flux vector are illustrated by a two-dimensional example. It is emphasized that considerable control can be exercised over the heat sources through the choice of the arbitrary functions and that the disadvantages of the inverse method can be further diminished through the use of a digital computer for numerical calculations.

Basic Equations. The governing equations for a steady, compressible, inviscid, perfect gas with zero heat conductivity, but containing heat sources, can be expressed as

$$\nabla \cdot (\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} + \nabla p = 0, \tag{2}$$

$$(\mathbf{V} \cdot \nabla) h_t = Q, \tag{3}$$

$$p = \rho R T, \tag{4}$$

where $h_t = C_p T + \frac{1}{2}\mathbf{V} \cdot \mathbf{V}$ is the total enthalpy. The heating function, Q, represents the heat added locally per unit mass of fluid and unit time; the other symbols, p, ρ, T, V, R and C_p are defined in the usual manner as the pressure, density, temperature, velocity, gas constant, and specific heat at constant pressure, respectively. The gas is also assumed to be calorically perfect (constant specific heats).

Equations (1)–(4) represent an underdetermined system if all the dependent variables are considered as unknowns. Thus, one of the dependent variables can be specified as a function of the other dependent variables or the independent variables. This is the reason why diabatic flows may be investigated theoretically by either the direct or the indirect approach depending on whether the heating function, Q, is specified or calculated, respectively.

The direct method would seem at first to be the more practical approach, since one has control over the heat source distribution and it is desired to know its effect on the flow field. However, there is some question as to the value of being able to specify precisely the heat source distribution, since for most practical applications it is difficult to know accurately just how and where the heat is being released in the flowing fluid. More precise information can be obtained experimentally about the flow pattern, pressure field and other measurable flow properties than about heat sources. Although direct control of the heat source distribution is lost, the indirect methods usually lead to the more simple mathematical treatment in that exactly linear equations which govern special classes of flow can be obtained without seriously restricting the magnitude of velocity, vorticity, or heating rate.

The common assumption of zero vorticity is quite a strong restriction on the possible heat source distributions since there is an important coupling between the vorticity, $\mathbf{\Omega} = \nabla \times \mathbf{V}$, and the heating effects. The relation (usually called Crocco's theorem)

$$\nabla h_t = \mathbf{V} \times \mathbf{\Omega} + T \nabla S \tag{5}$$

and the curl of this equation, where S is the specific entropy, reveals a great deal about the production of vorticity under various types of flow conditions. Vazsonyi [26] derived

Eq. (5) and discussed the theorems connected with it in detail; Tsien [5] and Prim [27] also add considerable to the interpretation of these results.

The general effect of the presence of heat sources along the streamlines is best displayed in the Crocco vector language (first used for flows having variable h_t [28] by Hicks and his co-workers). Hicks [23] derived the expression

$$\nabla \ln p_t = (1 - W^2)^{-1}(2\gamma/\gamma - 1)[\mathbf{W} \times \boldsymbol{\omega} - q\mathbf{W}] \tag{6}$$

relating the total pressure $p_t = p(1 - W^2)^{\gamma/1-\gamma}$, the heating coefficient $q = \frac{1}{2}\mathbf{W}\cdot\nabla\ln h_t$, the Crocco vector $\mathbf{W} = \mathbf{V}(2h_t)^{-1/2}$, and the rotationality of the $\mathbf{W}$ vector field $\boldsymbol{\omega} = \nabla\times\mathbf{W}$. Equation (6) can be regarded as a generalization of Crocco's theorem. The rate change of p_t along the streamlines is proportional to q, and the rate of change normal to the streamlines is proportional to ω.

An advantage of Hicks' inverse method [22], where the governing equations were expressed in terms of an irrotational vector parallel to the velocity vector, is that the velocity field is allowed to have restricted rotationality. Thus, the vorticity field can have sufficient generality to provide many possible modes of heat addition.

In Hicks' **N** theory, the integrability condition necessary for the equation of motion when the assumed irrotational vector $\mathbf{N} = \nabla\phi_N = (gRT)^{-1/2}\mathbf{V}$, $g = g(N)$, was used, gave rise to the integrability function $F(\phi_N)$. The quantities $F(\phi_N)$ and $g(N)$ were regarded as arbitrary functions and defined to specify the "form" and "character" of the flow, respectively. Dimsdale's transformation [14, 24, 25]

$$\Phi = \int \exp\left[-\frac{1}{2}\int F(\phi_N)\, d\phi_N\right] d\phi_N \tag{7}$$

then led to linear potential equations for three special choices of $F(\phi_N)$ when $g = 2(1 - N^2)^{-1}$.

Essentially the same approach is taken in this paper as was used in the **N** theory, but at some points more generality is introduced and retained until restrictions are applied to further simplify the theory. The significance of each restriction is explored in detail as various special flows are singled out from the general theory using the following unified approach.

Formulation of L Theory. The approach taken here is therefore characterized by the introduction of a vector **L**, parallel to the velocity vector **V**, and a heating coefficient q_L, proportional to the ratio of the heating function to the local specific enthalpy. The scalar function of proportionality is defined to be the same for both of these expressions.

$$\mathbf{L} = \sigma\mathbf{V}, \tag{8}$$

$$q_L = \sigma Q/C_pT. \tag{9}$$

The change in nomenclature from **N** to **L** is made to avoid confusion while comparing similar equations. The physical content of the resulting equations of course, is the same as that of the equations expressed in terms of the velocity, Mach, Crocco, or N vector language before restrictions are applied.

Equations (4), (8), and (9) will be used to eliminate ρ, **V**, and Q from Eqs. (1), (2), and (3) to obtain a new set of governing equations. First Eqs. (4) and (8) are inserted into (1) which is expanded to give the new continuity equation

$$\nabla\cdot\mathbf{L} + \mathbf{L}\cdot\nabla \ln p = \mathbf{L}\cdot\nabla \ln T + \mathbf{L}\cdot\nabla \ln \sigma, \tag{10}$$

then they are inserted into Eq. (2) and using the identity $(\mathbf{L}\cdot\nabla)\mathbf{L}=\nabla\frac{1}{2}L^2-\mathbf{L}\times(\nabla\times\mathbf{L})$, the equation of motion is written as

$$\sigma^2 RT\nabla \ln p + \nabla\tfrac{1}{2}L^2 = (\mathbf{L}\cdot\nabla \ln \sigma)\mathbf{L} + \mathbf{L} \times (\nabla \times L). \tag{11}$$

Equation (8) allows Eq. (3), the energy equation, to be expanded in the form

$$\sigma^3 Q + L^2(\mathbf{L}\cdot\nabla \ln \sigma) = \sigma^2 C_p \mathbf{L}\cdot\nabla T + \mathbf{L}\cdot\nabla\tfrac{1}{2}L^2. \tag{12}$$

Addition of Eqs. (11) and (12) after scalar multiplication of (11) by $\mathbf{L}$, insertion of (9), and use of $\gamma R = (\gamma - 1)C_p$ gives

$$q_L = \mathbf{L}\cdot\nabla \ln T + (1 - \gamma/\gamma)\mathbf{L}\cdot\nabla \ln p. \tag{13}$$

Equations (10), (11), and (13) form the basis for the work which follows and will now be simplified by a series of restrictions through which some new information can be obtained.

(*i*) *Restriction 1:*

$$\nabla \times \mathbf{L} = 0. \tag{14}$$

Only those flows in which $\nabla \times \mathbf{L}$ vanishes will be considered; such a "generalized irrotational flow" has been called "weakly rotational flow" [27, 29]. Assuming that the $\mathbf{L}$ vector field is irrotational allows the introduction of a potential function ϕ_L such that $\mathbf{L} = \nabla\phi_L$; a potential introduced in such a fashion is often referred to as the "quasi-potential" of the $\mathbf{V}$ field. This implies, in general, that the velocity field is rotational, the vorticity vector being given by

$$\mathbf{\Omega} = \nabla\sigma^{-1} \times \nabla\phi_L . \tag{15}$$

The functional form of σ obviously controls the vorticity distribution and also the mode of heat addition, since $\mathbf{\Omega}$ is connected with the thermodynamic variables through the Lamb vector, $\mathbf{\Omega} \times \mathbf{V}$, in (5). An irrotational velocity field results only if σ is conserved on equipotential surfaces.

One important consequence of (14) is that the velocity field is always "complex- or doubly-laminar", $\mathbf{\Omega}\cdot\mathbf{V} = 0$, a type of motion where $\mathbf{\Omega}$ is orthogonal to $\mathbf{V}$. Another conclusion can be drawn from Eq. (15) by noting that $\mathbf{\Omega}\cdot\nabla\sigma = 0$; this indicates that the vortex lines must coincide with the level surfaces of σ.

(*ii*) *Restriction 2:*

$$\sigma = \alpha(p)\beta(L)\delta(T). \tag{16}$$

In general, one might consider σ to be a function of any or all of the dependent variables and the independent variables. However, from the formulation of the $\mathbf{N}$ theory, it appears that $\sigma = \sigma(p, L, T)$ is the most useful form. The advantage of this functional form will be clear after Restriction 5 is discussed, although the first indication of its usefulness was obtained from the special cases treated by Dimsdale's transformation [24, 25]. Allowing σ to depend on p is essentially what distinguishes this approach from the $\mathbf{N}$ formulation.

In particular, it will be assumed in what follows that σ is a separable function of p, L, and T such that

$$\nabla\phi_L = \alpha\beta\delta\mathbf{V} \tag{17}$$

and therefore

$$\mathbf{\Omega} = \mathbf{V} \times \nabla \ln \alpha + \mathbf{V} \times \nabla \ln \beta + \mathbf{V} \times \nabla \ln \delta, \tag{18}$$

where α, β, and δ are arbitrary functions as indicated in Eq. (16) and each can be chosen at different stages in the development of the theory.

(*iii*) *Restriction 3:*

$$\delta = (RT)^{-1/2}. \tag{19}$$

From past experience, one regards determination of the temperature throughout the flow field as a formidable task. In general, numerical methods are necessary to integrate the temperature equation along the streamlines; it is for this reason that it is wise to choose δ so as to uncouple the equation of motion from explicit temperature effects. This is accomplished if δ is taken in the form of Eq. (19) with dimensions of reciprocal velocity such that the first integral of the equation of motion is not difficult to attain. Use of Eqs. (14), (16), and (19) allows Eq. (11) to be written

$$\alpha^2 \nabla \ln p + \beta^{-2} \nabla \tfrac{1}{2} L^2 = [\beta^{-2} \mathbf{L} \cdot \nabla \ln \sigma] \nabla \phi_L . \tag{20}$$

(*iv*) *Restriction 4*:

$$\beta^{-2} \mathbf{L} \cdot \nabla \ln \sigma = F(\phi_L). \tag{21}$$

Integration of Eq. (20) depends on satisfaction of the integrability condition expressed by (21) where $F(\phi_L)$ will be referred to as an integrability function.

Taking the scalar product of (20) with the differential of the local position vector then allows its integration

$$\int \alpha^2 \, d \ln p + \int \beta^{-2} \, d\tfrac{1}{2} L^2 = \int F(\phi_L) \, d\phi_L + P_0 , \tag{22}$$

where P_0 is a constant. Use of the integrability function to eliminate T from Eq. (10) allows that equation to be written in the form

$$\nabla \cdot \mathbf{L} = \mathbf{L} \cdot \nabla \ln (\alpha^2/p) - \beta^2 F(\phi_L) + (d \ln \beta^2/dL) \mathbf{L} \cdot \nabla L, \tag{23}$$

where $(d\beta/dL)\mathbf{L} \cdot \nabla L$ has been substituted for $\mathbf{L} \cdot \nabla \beta$.

$F(\phi_L)$ is regarded as an additional arbitrary function to be specified, and (21) is utilized as the relationship necessary to make (13), (22), and (23) a determinant system in the variables ϕ_L , p, T, and q_L . It is obvious that the choice of $F(\phi_L)$ has a strong influence on the nature of the motion to be studied.

(*v*) *Restriction 5*:

$$\alpha = (p/\tau_0)^{1/2}. \tag{24}$$

At this point it should be noted that while α is an arbitrary function of p, the $\mathbf{L}$ theory becomes the $\mathbf{N}$ theory for $p = p(L)$, which is seen to be the case if $F(\phi_L) = 0$, or if $\phi_L = \phi_L(L)$. On the other hand, we see that α being constant, or more specifically $\alpha = 1$, leads to the $\mathbf{N}$ theory for any $F(\phi_L)$ which in turn can be reduced to the Crocco or Mach vector languages by the proper choice of β; thus the formulation given here is more general than the $\mathbf{N}$ theory when α is arbitrary.

We will choose α in the form of Eq. (24) for the remainder of this paper, because this choice simplifies not only Eq. (23) but also leads to a new and simple form of all the governing equations for weakly rotational diabatic flow. The symbol τ_0 denoting a reference dynamic pressure, $\frac{1}{2}\rho_0 V_0^2$, is introduced so that L and β may be taken to be dimensionless quantities. Therefore, the functions q_L and $F(\phi_L)$ have dimensions of reciprocal length, whereas ϕ_L has dimensions of length and σ has dimensions of reciprocal velocity.

The unexpected result that the $\alpha = 1$ and $\alpha = (p/\tau_0)^{1/2}$ theories are related for flows other than the simple ones with $p = p(L)$ can be established through the use of Eq. (7) with $F(\phi_L) = -F(\phi_N)$, and Eq. (22) with $\alpha = 1$. In addition, Dimsdale's transformation can be generalized to include all cases where the discriminant of the potential equation is of constant value $(\epsilon - 1)$, except $\epsilon = 2$, by replacing the $\frac{1}{2}$ in Eq. (7) by $(2 - \epsilon)^{-1}$. The result [4] is that Dimsdale's transformation is essentially effected on the $\alpha = 1$ equations by Restriction 5 for arbitrary $F(\phi_L)$ when the discriminant of (23) is a constant, not unity. Although it appeared [4] that the $\alpha = (p/\tau_0)^{1/2}$ case was entirely contained in the $\alpha = 1$ equations, closer examination shows that there are many classes of flow governed by each theory which are not contained in the other.

Significance of the choice of $\beta(L)$ and $F(\phi_L)$. Briefly, the problem now formulates as follows: The continuity equation governs the potential function and the momentum equation has been integrated

$$\nabla^2\phi_L = \tfrac{1}{2}(1 - D)\nabla\phi_L \cdot \nabla \ln (\nabla\phi_L)^2 - \beta^2 F(\phi_L), \tag{25}$$

$$p + \tau_0 \int \beta^{-2}\, d(\tfrac{1}{2}L^2) = P(\phi_L), \tag{26}$$

where $P(\phi_L) = P_0 + \tau_0 \int F(\phi_L) d\phi_L$ and $D = (1 - d \ln \beta^2 / d \ln L)$ involve the two remaining arbitrary functions. Following the precedent already set [22], $\beta(L)$ and $F(\phi_L)$ are defined to specify the "character" and "form" of the flow, respectively. A "class" of diabatic flows is specified when both the "character" and the "form" have been chosen.

Thus, using a potential function satisfying Eq. (25), the pressure is found from Eq. (26), and the energy equation

$$q_L = (M^2 - 2)\beta^2 F(\phi_L) - (M^2 + D - 1)\tfrac{1}{2}\nabla\phi_L \cdot \nabla \ln (\nabla\phi_L)^2 \tag{27}$$

becomes the formula for the heating coefficient, where M is the local Mach number, $\gamma M^2 = \tau_0 L^2/p\beta^2$. The density or temperature is supplied along streamlines by the integrability condition

$$\nabla\phi_L \cdot \nabla \ln \rho = 2\beta^2 F(\phi_L) - \nabla\phi_L \cdot \nabla \ln \beta^2 : \tag{28}$$

The velocity and the heating function are then obtained from (8) and (9), and the vorticity is given by

$$\mathbf{\Omega} = \tfrac{1}{2}\mathbf{V} \times \nabla \ln \rho + \mathbf{V} \times \nabla \ln \beta. \tag{29}$$

The only difficulty in obtaining the entire solution is the integration of (25) with appropriate boundary conditions. Once ϕ_L is known, (28) is a linear first-order partial differential equation in $\ln \rho$, and it can be solved analytically or numerically integrated along the streamlines.

From

$$D = \left(1 - \frac{d \ln \beta^2}{d \ln L}\right) \tag{30}$$

it is obvious that the type of the potential equation is determined solely by the choice of β, since D represents the negative of the discriminant of (25); hence $D > 0, = 0, < 0$ result in elliptic, parabolic and hyperbolic types, respectively. The choice of $F(\phi_L)$ has no effect on the type of the potential equation, but it has a powerful influence on the flow properties as is shown by Eqs. (25)–(29). We note that when β is constant, $D = 1$, and the second-order, non-linear term is eliminated from (25), giving some flows whose "character" will be investigated in greater detail later in this work. These flows correspond to those of the **N** theory when $g = 2(1 - N^2)^{-1}$.

Although Eqs. (25)–(29) are written in their preferred form for flow calculations, the significance of the various terms in the new equations is best indicated when three mutually orthogonal unit vectors ($\mathbf{s}$, $\mathbf{n}$, $\mathbf{t}$) are introduced. Due to Restriction 1, the vorticity, Lamb, and $\mathbf{L}$ vectors conveniently form a mutually orthogonal set of vectors so that $\mathbf{L} = L\mathbf{s}$, $\mathbf{\Omega} = \Omega\mathbf{t}$, and $\mathbf{\Omega} \times \mathbf{L} = \Omega L\mathbf{n}$. In addition the notation

$$\partial/\partial s = \mathbf{s}\cdot\nabla, \qquad \partial/\partial n = \mathbf{n}\cdot\nabla, \qquad \partial/\partial t = \mathbf{t}\cdot\nabla \tag{31}$$

for the directional derivatives is used, denoting spatial differentiation parallel to the streamlines, the Lamb vector lines, and the vortex lines, respectively.

The restrictions applied thus far limit the variation of σ such that

$$\frac{\partial}{\partial s}\ln\sigma = \beta^2F(\phi_L)/L, \qquad \frac{\partial}{\partial n}\ln\sigma = \Omega/V, \qquad \frac{\partial}{\partial t}\ln\sigma = 0, \tag{32}$$

where $\sigma = (\rho/\tau_0)^{1/2}\beta$. The s-component of the equation of motion can be expressed as

$$\frac{\partial}{\partial s}\ln p + \gamma M^2\frac{\partial}{\partial s}\ln L = \gamma M^2\beta^2F(\phi_L)/L, \tag{33}$$

where M is the local Mach number and $L^2 = \beta^2\rho V^2/\tau_0$. The continuity equation

$$\nabla\cdot\mathbf{s} + D\frac{\partial}{\partial s}\ln L + \beta^2F(\phi_L)/L = 0 \tag{34}$$

relates $\nabla\cdot\mathbf{s}$ to β, L, and $F(\phi_L)$. Equation (34) is more simple than its counterpart in other vector languages. It has been found that $\nabla\cdot\mathbf{s}$ can be interpreted as the fractional rate of change of stream-tube area with respect to arc length along the stream-tube [28]. If β is taken to be $L^{1/2}$, $D = 0$, and $F(\phi_L)$ completely controls the stream-tube area variation; thus, for $F(\phi_L) = 0$, only constant area stream-tube flow is possible. This is the same result obtained from the **N** theory parabolic case [22]. The energy equation

$$q_L/L = \nabla\cdot\mathbf{s} + (1 - M^2)\left[\frac{\partial}{\partial s}\ln L - \beta^2F(\phi_L)/L\right] \tag{35}$$

relates the heat sources to the behavior of the functions appearing in Eqs. (33) and (34). The well known result that the streamlines must be diverging if heat is being added continuously at the sonic condition appears as a special case from this equation.

Equation (34) can be used to eliminate any one of the three terms on the right of Eq. (35) to get three alternate forms for the energy equation, one of which is equivalent to Eq. (27). Another can be written

$$q_L/L = (2 - M^2)\nabla\cdot\mathbf{s} + (1 - M^2)(D + 1)\frac{\partial}{\partial s}\ln L. \tag{36}$$

The coefficient of $\nabla \cdot \mathbf{s}$ changes sign at $M^2 = 2$, and that of $\partial/\partial s \ln L$ at $M^2 = 1$ and $D = -1$. A rather interesting case arises when $\beta = L$, $D = -1$, as the heat sources are then proportional to the rate of area change and change sign at $M^2 = 2$. This special case is distinguished by the fact that the dynamic pressure, $\frac{1}{2}\rho V^2$, is constant. The analogous situation, with the coefficient of the integrability function vanishing identically, occurs in the **N** theory for constant Mach number flow.

When the heat sources are related to $\nabla \cdot \mathbf{s}$ and $F(\phi_L)$

$$Dq_L/L = (M^2 + D - 1)\nabla \cdot \mathbf{s} + (M^2 - 1)(D + 1)\beta^2 F(\phi_L)/L \tag{37}$$

and we observe another, more general, case in which parallel flow is iso-energetic; if $F(\phi_L)$ is chosen identically zero, the heat sources not only change sign locally at minimum area but also at $M^2 = 1 - D$. This "form" of flow has possible application for external aerodynamics problems due to the absence of heating at infinity. Several other properties of the $F(\phi_L) = 0$ "form" of flow will be given later when it is discussed in detail.

It is noted that the coefficients in the alternate forms of the energy equation emphasize the importance of $M^2 = 1, 2, 1 - D$ in determining the nature of the heat sources. Also, the most interesting cases are obtained when the value of the discriminant of Eq. (25) remains constant because new simplifications can result. Examples which illustrate these simplifications have been discussed briefly for $D = -1, 0, +1$. More generally, it can be shown that the choice $\beta = L^{\epsilon/2}$ gives $D = 1 - \epsilon$ where ϵ has any constant value. Some enlightening equations expressing the local pressure's dependence on ϵ, ϕ_L, M, and the dynamic pressure have been studied for this case [4].

Two special cases which warrant further investigation have been mentioned in the preceding discussion. Before giving a detailed discussion of the flows "characterized" by $\beta = 1$, the $F(\phi_L) = 0$ "form" of flow will be studied.

$F(\phi_L) = 0$, relation to irrotational V flows. In this case a useful approach to the solution of (25) in two dimensions is the independent variable transformation from the x, y-plane to the ξ, η-plane, where $\mathbf{L} = \xi \mathbf{i} + \eta \mathbf{j}$. This interchange of dependent and independent variables is valid if the Jacobian of the transformation does not vanish. The resulting equation can be written in final form using Legendre's transformation [30]

$$\phi_L = \xi x + \eta y - \Phi_L, \qquad x = \frac{\partial \Phi_L}{\partial \xi}, \qquad y = \frac{\partial \Phi_L}{\partial \eta}. \tag{38}$$

Thus, the potential equation in terms of the Legendre transform Φ_L, of the potential ϕ_L, becomes

$$(\xi^2 + D\eta^2)\frac{\partial^2 \Phi_L}{\partial \xi^2} + 2(1 - D)\xi\eta \frac{\partial^2 \Phi_L}{\partial \xi\, \partial \eta} + (D\xi^2 + \eta^2)\frac{\partial^2 \Phi_L}{\partial \eta^2} = 0, \tag{39}$$

where $F(\phi_L)$ has been taken to be zero; under such a restriction Eq. (39) takes the form of a homogeneous, second-order, linear partial differential equation with $\beta(L)$ still unspecified. The ξ, η-plane is not the hodograph plane except when σ is constant, although the similarities would lead one to regard it as a "generalized hodograph plane"; the problems encountered here in the transfer of a solution in the ξ, η-plane back to the physical plane are much the same as those encountered when using the hodograph method.

In view of these results it is interesting to note that when $F(\phi_L) = 0$, comparison of the **L** equations and those of irrotational **V** flow [23] leads to a distinct correspondence. In fact, one can observe the following relationships

$$\begin{aligned} \mathbf{V}^* &= V_0 \nabla \phi_L\,, \qquad \rho^* = \rho_0 \beta^{-2}, \qquad p^* = p, \\ T^* &= \beta^2 T \rho/\rho_0\,, \qquad M^* = M, \qquad Q^* = \beta^3 Q(\rho/\rho_0)^{3/2}, \end{aligned} \tag{40}$$

where the asterisk denotes the flow properties of irrotational **V** flows. These relations can also be obtained using a so-called "substitution principle" since σ is a scalar function which in this case remains constant along the streamlines. Equations (40) can be considered a similarity law which relates this "form" of rotational, compressible, diabatic flow to irrotational flow having the same streamlines, pressure field and Mach number distribution, with appropriate modifications being made to the other flow properties. It is obvious that the **L** formulation possesses no particular advantage over the **V** language when $F(\phi_L) = 0$, since any "weakly rotational solution" obtained from the **L** theory when $F(\phi_L) = 0$ can be constructed from the appropriate "irrotational solution" obtained from the **V** theory. Unfortunately there is no easy method to generate irrotational **V** solutions at present. However, when a solution, ϕ_L, to the potential equation is obtained for $F(\phi_L) = 0$, the flow properties ρ, T, V, and Q are easily found because σ is conserved on stream surfaces composed of intersecting families of vortex lines and streamlines.

$\beta = 1$, elliptic flow. It is obvious that the governing equations are simplified a great deal for the special case of $\epsilon = 0$, which implies $\beta = 1$ and $D = 1$. In this instance, it should be noted that the same potential equation is obtained if $d\beta/dL \neq 0$ and the last term in (23) vanishes indicating that L is conserved along the streamlines, thus allowing β to vary from streamline to streamline; however, choosing β to be constant throughout the flow field does not place such a restriction on L. Hence, β is taken to be unity here, as this is considered to be the more general of the two alternatives. Since it was pointed out previously that Dimsdale's transformation is effected by Restriction 5 when $\beta = 1$, one can expect that the linear, transformed **N** equations [14, 24, 25], can now be obtained for special choices of $F(\phi_L)$.

When $\beta = 1$, $\mathbf{L} = (\rho/\tau_0)^{1/2}\mathbf{V}$, and Eqs. (25)–(29) become

$$\nabla^2 \phi_L + F(\phi_L) = 0, \tag{41}$$

$$P(\phi_L) = p + \tfrac{1}{2}\rho V^2, \tag{42}$$

$$q_L = (M^2 - 2)F(\phi_L) - \tfrac{1}{2}M^2 \mathbf{L} \cdot \nabla \ln L^2, \tag{43}$$

$$\mathbf{L} \cdot \nabla \ln \rho = 2F(\phi_L), \tag{44}$$

$$\mathbf{\Omega} = \tfrac{1}{2}\mathbf{V} \times \nabla \ln \rho, \tag{45}$$

which can also be obtained directly from Eqs. (1)–(4) using only the assumption that $\rho^{1/2}\mathbf{V}$ is irrotational; however, if this restriction had been applied at the outset, the orientation provided by the framework of the general theory would have been lost. The basic question as to the applicability of this single assumption to a physical problem is best answered by stating that these equations apply only to flows in which the vorticity follows the expression given by (45); that is, the vorticity distribution depends on the magnitudes of ρ, $\mathbf{V}$, $\nabla\rho$, and the direction of $\nabla\rho$ relative to that of $\mathbf{V}$.

A vector defined similarly to **L** has been used briefly before in an investigation of adiabatic flow to establish the so-called "substitution principle" and it was called the "kinetic flux vector" [31]. This name supposedly refers to the fact that $(\tau_0/2)L^2$ represents the kinetic energy of a unit volume of fluid. The simplifications occurring here are obviously connected with those obtained by Prim [31], however in that case the use of this vector did not lead to any unique results since it was used in an alternate derivation of a principle originally developed by another technique.

A physical interpretation of $F(\phi_L)$ can now be attained from the potential equation using the divergence theorem of Gauss. Since $\nabla \cdot \mathbf{L}$ represents the excess of L per unit volume and time leaving a point in the field over that arriving, $F(\phi_L)$ determines the nature of the sources of $\rho^{1/2}V$ throughout the flow. Hence the sign and strength of the sources of $\rho^{1/2}V$ are constant on equipotential surfaces.

The simplicitly of Eqs. (41)–(45) allows diabatic flows with irrotational $\rho^{1/2}\mathbf{V}$ to be investigated in greater detail than the more general flows [4].

Flow field calculations for $\beta = 1$. Relatively few exact solutions for elementary diabatic flow problems have been presented. One-dimensional, constant-area, diabatic flow studies have yielded much of the fundamental knowledge which we now possess about the effects of heat addition [7–10]. The one-dimensional radial and vortex flows, which were calculated using the Crocco vector language [14], illustrated several additional differences between adiabatic and diabatic compressible flows, such as absence of limit circles and non-minimal stream-tube area at sonic velocity.

The two-dimensional examples treated through the **N** language using Dimsdale's transformation [24, 25] demonstrated that limit curves, associated with zero absolute pressure and infinite local Mach number, are often present in diabatic flows calculated by the inverse method. Physically, at such limit curves the local velocity has reached its limiting value of $(2h_t)^{1/2}$, corresponding to zero absolute static temperature. These limit curves can be avoided then by not extracting too much heat in supersonic regions. Unfortunately, the inverse method being used here makes it difficult to avoid limit curves in all cases, since direct control over the heat sources has not been retained. Actually, since problems which involve heat addition are the more common, those with heat sinks are not so interesting in any event. The behavior discussed above is somewhat different than that which is observed when limit lines, usually associated with the breakdown of the hodograph method, occur in adiabatic potential flow.

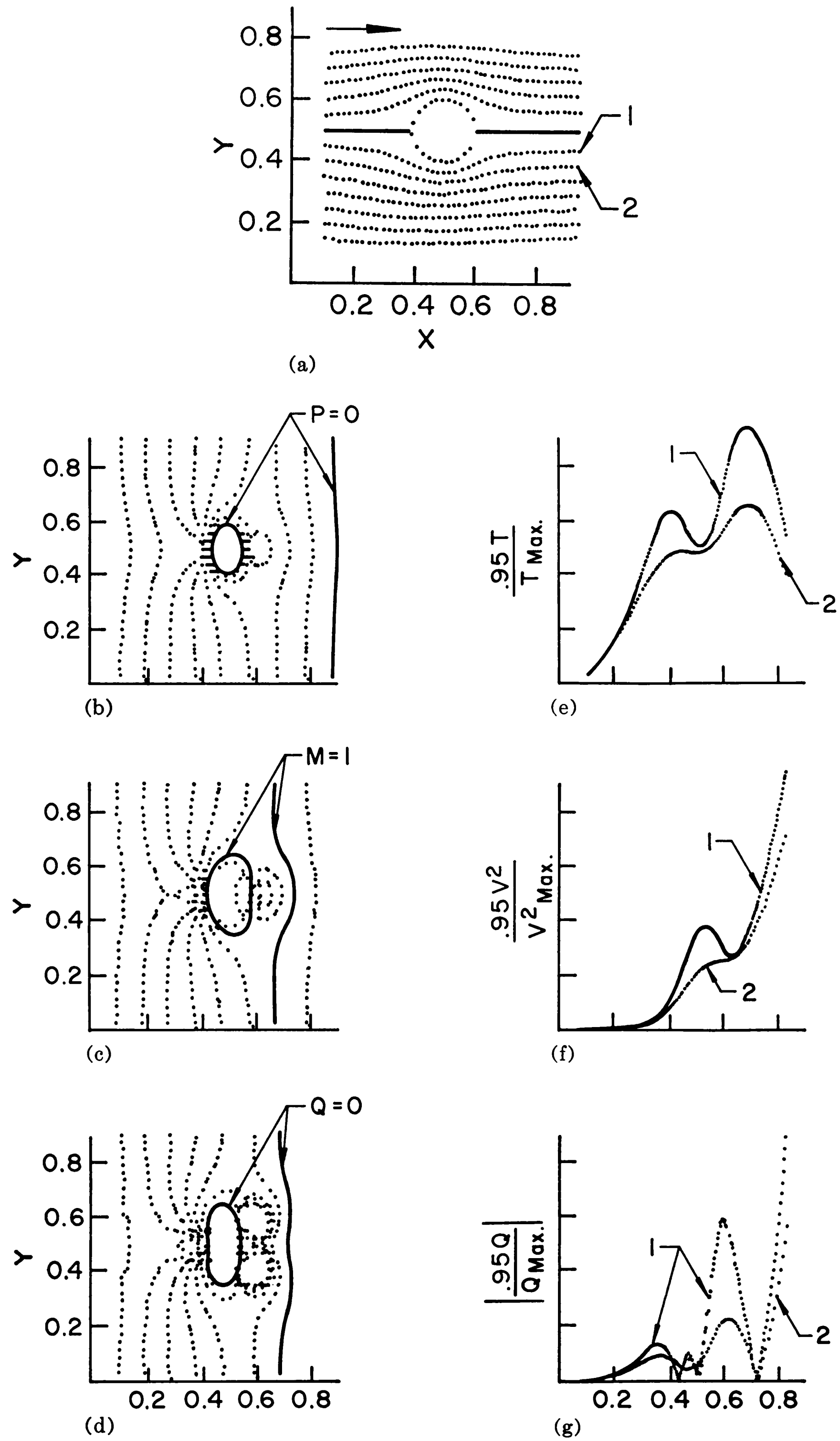
The fundamental solutions which were investigated in considerable detail [24, 25] corresponded to taking the equipotential curves to be the conic sections. It did not seem possible to integrate the temperature equation in general form at that time, therefore the velocity and heating function were not obtained throughout the flow fields. Now because of the simplicity of the equations in the **L** language when $\beta = 1$, the general solutions for the density variation in these basic flow fields are easily found. Thus if $\tau_0^{1/2}\phi_L = ax^2 + ey$, then $\nabla\phi_L \cdot \nabla \ln \rho = -4a$, and

$$\ln \rho = f(\psi) - 4ay/e, \qquad e \neq 0, \tag{46}$$

where constant values of $\psi = x \exp(-2ay/e)$ label the streamlines. Similarly, if $\tau_0^{1/2}\phi_L = ax^2 + cy^2$, then $\nabla\phi_L \cdot \nabla \ln \rho = -4(a + c)$, and

$$\ln \rho = f(\psi) - \left(\frac{c}{a} + 1\right) \ln x^2, \qquad a \neq 0, \tag{47}$$

where $\psi = y^{-1}x^{c/a}$, and a, e, and c are constants. Therefore using the equation of state,


0.8
0.6
Y
0.4
0.2
0.2 0.4 0.6 0.8
X
1
2
(a)
P=0
M=1
Q=0
Y
0.8
0.6
0.4
0.2
0.2 0.4 0.6 0.8
(b)
(c)
(d)
.95 T / T Max.
.95 V2 / V2 Max.
|.95Q / Q Max.|
(e)
(f)
(g)

the pressure equation, and the equation for the heating coefficient, which have already been given explicitly [24, 25], the remainder of the flow properties can be obtained for these basic flows.

The basic flows just discussed give $F(\phi_L) = -2(a + c)\tau_0^{-1/2}$, a constant. However, the least complex "class" of diabatic flows obtainable from the **L** theory results from the combined simplicity of the $F(\phi_L) = 0$ "form" of flows and those "characterized" by $\beta = 1$. Due to the fact that when a density gradient is present, it must be normal to the streamlines, such flows are often called "inhomogeneous incompressible flows" [29]. Also, since the heating rate is proportional to Mach number and rate of area change, no heating is present at stagnation points or when parallel flow occurs. Two-dimensional examples of this "class" can be treated within the framework of complex variables since their complete description can be written immediately when ϕ_L is given as any harmonic function. Examples of these flows have been given [4], and are of interest primarily because of the possibility of superposing them on other basic diabatic flows to create more general flow fields.

A single two-dimensional example is presented in this section to illustrate the nature of flows with irrotational kinetic flux vector. This example corresponds to streamwise-nonuniform parallel flow ($c/a = 0$), disturbed by a doublet, $\nu_0(x - x_0)/r_0^2$, superposed through $\tau_0^{1/2}\phi_L$. Since the doublet alone does not create a closed body in this case, a sink, $-\frac{1}{2}\mu_0 \ln r_0^2$, where $r_0^2 = (x - x_0)^2 + (y - y_0)^2$, was also superposed and μ_0 was adjusted until a cylindrical shaped body was obtained. Parameters ν_0 and μ_0 are constants, and x_0 and y_0 are constant coordinates of the singular point. This example is a special case selected from a more general parameterized example [4], for which two main computer programs were written for Illiac (Illinois Automatic Computer). The results from both programs in the form of plots on a cathode ray tube were recorded on 35mm film. One of the programs produces isoline plots from a square grid of values of the functions p, M^2 and q_L, thus displaying the "critical curves" (limit curve for zero pressure, sonic line, zero rate of heat addition curve). The other computer program is concerned with the numerical integration of Eq. (44) for $\rho(x, y)$ and calculating flow properties T, V^2 and $|Q|$ along given streamlines.

The results from this case are presented as Figs. 1a–g. Figure 1a is the flow pattern and Figs. 1b–d represent isoline plots for p, M, and q_L, respectively. The isoline levels are not needed here but are given elsewhere [4]; however the critical curves are labeled on the plots. Figures 1e–g are plots of T, V^2, and $|Q|$, respectively, along two typical streamlines. It is noted that the ordinate on these plots represents a normalized value of the function as shown on the plots. The function is plotted versus the x coordinate along a given streamline and the y coordinate must be obtained from the corresponding streamline in the ψ-plot. It should also be pointed out that in the Q plot the sign is not preserved, so that the sign of Q must be obtained from other considerations, i.e., the isoline plot

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FIG. 1. Flow field for doublet and sink in streamwise-nonuniform parallel flow.
a. Flow pattern.
b. Pressure isolines.
c. Mach number isolines.
d. Heating coefficient isolines.
e. Temperature variation along streamlines.
f. Velocity variation along streamlines.
g. Rate of heat addition along streamlines.

for q_L. For this reason only positive and zero q_L levels are displayed here so that the regions of heat sinks and heat sources can be identified. The constants γ, P_0, x_0 and y_0 were taken to be 1.4, 10^{-2}, 0.5 and 0.5, respectively. When $a = 0.0555$, $\nu_0/2ax_0 = 0.1$ and $\mu_0 = 0.000595$, it is observed (see Fig. 1a) that the body shape is nearly cylindrical and of approximate radius $r_b = (\nu_0/2ax_0)^{1/2}$. The direction of the undisturbed flow is taken in the positive x-direction. As a result of the pressure gradient the body experiences a force per unit length in the x-direction of approximately $10\pi a^2 x_0 r_b^2$.

In the basic undisturbed flow, heat is added up to $M = 1$ ($x = 0.688$) and then extracted until the limit curve occurs. That is, the flow is accelerated from zero velocity until zero pressure is attained. The density drops off along the streamlines as x^{-2} in the undisturbed flow [see (47)] and it is observed that there is little change when a body is present. However, it can be seen that the other flow properties are affected a great deal by the disturbance (see Figs. 1b–g). Several stationary values appear in the variation of the temperature, velocity, and heating rate along the streamlines (see Figs. 1e–g); the absolute maximum temperature may appear either upstream or downstream of the body depending on the location of the body given by x_0.

Very high gradients in the rate of heat addition and other flow properties occur in the vicinity of the singular point, in particular just downstream of it; this can be seen from the isoline plots (Figs. 1b–d) since the isolines become quite dense in these regions. It is possible to have the body immersed in a region with only heat sources present in the immediate vicinity of the body, if the disturbance is located in the low velocity portion of the basic flow field. In addition, one can expect a limit curve to appear outside the closed body for any x_0 greater than approximately 0.568, which has a nominal undisturbed local Mach number of 0.6 associated with it. Additional calculations for various values of the parameters for this example and others have also been given [4].

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