FORMAL EQUIVALENCE OF THE NONLINEAR STRING AND ONE-DIMENSIONAL FLUID FLOW*

By GERALD ROSEN (*Southwest Research Institute, San Antonio, Texas*)

Summary. By applying transformations to the dynamical equation for the longitudinal vibrations of a nonlinear model string, we obtain a set of equations which describes the one-dimensional flow of an ideal compressible polytropic fluid. Thus it is shown that the nonlinear string problem is formally equivalent to the classical problem of fluid flow analyzed by Riemann and by others.

Proof of the equivalence. The nonlinear wave equation

$$y_{tt} = (1 + \epsilon y_z)^\alpha y_{zz} \tag{1}$$

with constant α and ϵ governs the longitudinal vibrations of a certain model string, according to Zabusky and Kruskal [1], who have presented an analysis of the Cauchy problem for (1). We shall show in this note that (1) expresses the exact mathematical content of the set of three equations for the one-dimensional flow of an ideal compressible polytropic fluid, namely, the set of equations [2]

$$\begin{aligned} \rho_t + u\rho_z + \rho u_z &= 0, \\ \rho u_t + \rho u u_z + p_z &= 0, \\ (p/\rho^k) &= \text{const.} \end{aligned} \tag{2}$$

where ρ is the fluid density, p the fluid pressure, u the fluid velocity, k the polytropic index (a constant parameter), and t and z are the independent variables representing time and distance. Consequently, the Cauchy problem for (1) and the Cauchy problem for the set (2) are equivalent mathematical problems.

To prove our assertion, we first relate a fluid density ρ to the string deflection y in (1) by setting

$$\frac{\rho_0}{\rho} \equiv 1 + \epsilon y_z , \tag{3}$$

where ρ_0 is a physical constant having the dimensions of density. Using (1), we obtain

$$\left(\frac{\rho_0}{\rho}\right)_{tt} = \frac{1}{(\alpha + 1)} \left[\left(\frac{\rho_0}{\rho}\right)^{\alpha+1}\right]_{zz} . \tag{4}$$

Next, we introduce a polytropic index k and a fluid pressure p by setting

$$k \equiv -(\alpha + 1), \quad (5) \qquad p/\rho^k = p_0/\rho_0^k , \quad (6)$$

where p_0 is a physical constant having the dimensions of pressure. Equation (4) now becomes

$$\left(\frac{\rho_0}{\rho}\right)_{tt} + \left(\frac{p}{kp_0}\right)_{zz} = 0, \tag{7}$$

*Received December 4, 1964.

which can be viewed as an integrability condition [3] that guarantees the existence of a fluid velocity u satisfying

$$[\rho^{-1}(k\rho_0 p_0)^{1/2}]_t - u_x = 0, \tag{8}$$

$$u_t + [p/(k\rho_0 p_0)^{1/2}]_x = 0. \tag{9}$$

Moreover, Eqs. (8) and (9) can be viewed as integrability conditions that guarantee the existence of quantities z and ϕ satisfying

$$dz = [(k\rho_0 p_0)^{1/2}/\rho]\, dx + u\, dt, \tag{10}$$

$$d\phi = (k\rho_0 p_0)^{1/2} u\, dx - p\, dt, \tag{11}$$

differential conditions which imply that

$$(k\rho_0 p_0)^{1/2}\, dx = \rho\, dz - \rho u\, dt, \tag{12}$$

$$d\phi = \rho u\, dz - (\rho u^2 + p)\, dt. \tag{13}$$

Thus, if ρ, u, and p are considered as functions of t and z (the latter interpreted as a space coordinate), Eqs. (12) and (13) yield

$$\rho_t + (\rho u)_z = 0, \tag{14}$$

$$(\rho u)_t + (\rho u^2 + p)_z = 0. \tag{15}$$

Eqs. (6), (14), and (15) are equivalent to the set of Eqs. (2). Hence, Eq. (1) expresses the exact mathematical content of the set of Eqs. (2), and therefore classical results for the latter set of equations provide an immediate solution for the Cauchy problem for Eq. (1). In particular, the development in time of discontinuities in y_x for $\alpha \neq 0$ follows from Riemann's theory of shock formation [4], that is, the development in time of discontinuities in ρ for $k \neq -1$, by evoking the formal correspondence given by Eq. (3).

References

1. N. J. Zabusky, J. Math. Phys. **3**, 1028 (1962)
M. D. Kruskal and N. J. Zabusky, J. Math. Phys. **5**, 231 (1964)
2. For example, see: R. von Mises, *Mathematical theory of compressible fluid flow*, Academic Press Inc., New York, 1958, Article 12, p. 155
3. A more general form of Eq. (7), a form that takes account of real fluid viscosity, has been derived and used as a starting point for non-Riemannian fluid flow theory; see: G. Rosen, Phys. of Fluids **2**, 517 (1959); **3**, 188 (1960); **3**, 191 (1960)
4. A modern analysis of this classical problem is presented by: P. A. Fox, J. Math. and Phys. **34**, 133 (1955)