QUARTERLY OF APPLIED MATHEMATICS

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The Quarterly prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts (one copy only) submitted for publication in the Quarterly of Applied Mathematics should be sent to Professor E. T. Onat, Managing Editor, or Miss Marjorie B. Luther, Editorial Assistant, Editorial Office, Box F, Brown University, Providence, R. I. 02912, either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only. The authors' institution will be requested to pay a publication charge of $5.00 per page which, if honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter t and the prime (′), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set above letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which follow the letter should be used.

Square roots should be written with the exponent ½ rather than with the sign √.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponents with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus, \( \exp \left( a^2 + b^3 \right)^{1/2} \) is preferable to \( e^{(a^2 + b^3)^{1/2}} \).

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

\[
\frac{\cos \left( \frac{\pi}{2b} \right)}{\cos \left( \frac{\pi a}{2b} \right)}
\]

In many instances the use of negative exponents permits saving of space. Thus,

\[
\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.
\]

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

\[
(a + bx) \cos t \text{ is preferable to } \cos t (a + bx).
\]

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

\[
\{a + (b + cz)^n\} \cos k y \text{ is preferable to } ((a + (b + cz)^n) \cos k y^n).
\]

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (for books—S. Timoshenko, Strength of materials, vol. 2, Macmillan and Co., London 1931, p. 371; for periodicals—Lord Rayleigh, On the flow of viscous liquids, especially in three dimensions, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, On the flow of viscous fluids is preferable to On the Flow of Viscous Fluids, but the corresponding German title would have to be rendered as Über die Strömung schwer Flüssigkeiten.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.,” even if this special abbreviation is defined somewhere in the text.
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Series translated and edited by Richard A. Silverman ...

SPECIAL FUNCTIONS AND THEIR APPLICATIONS
by N. N. Lebedev, Physico-Technical Institute of the Academy of Sciences of the U.S.S.R.
A self-contained discussion of the complex variable techniques, this volume emphasizes applications to the boundary value problems of mathematical physics, but is suitable for students with advanced calculus background. March 1965, 320 pp., $12.00

PROBLEMS OF MATHEMATICAL PHYSICS
by N. N. Lebedev, I. P. Skalskaya and Y. S. Ufyand, all Members of the Physico-Technical Institute of the Academy of Sciences, U.S.S.R.
This text presents a systematic exploration of the superposition method and its ramifications, as applied to problems of mechanics, heat conduction and electromagnetic theory. September 1965, 448 pp., $12.00

INTEGRAL, MEASURE AND DERIVATIVE: A Unified Approach
by G. E. Shilov, Moscow State University and B. L. Gurevich, noted Russian author
Outstanding features of this book include: Unified systematic exploitation of the Daniell approach, lucid expository style, an abundance of carefully selected problems, and an up-to-date bibliography. 1966, Price and publication date to be announced.

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE, VOLUME I
by A. I. Markushevich, Moscow State University
A definitive and detailed treatment of analytic functions theory, emphasizing those aspects of the subject of greatest practical and theoretical significance. January 1965, 480 pp., $12.00

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE, VOLUME II
by A. I. Markushevich, Moscow State University
A new text for the basic senior or first-year graduate course in complex variable theory; for all mathematicians, physicists and engineers. August 1965, 333 pp., $12.00

for approval copies, write: Box 903 PRENTICE-HALL, Englewood Cliffs, N. J.
BOOK REVIEWS


Differential equations with periodic coefficients occur in the course of many investigations, particularly in connection with the stability of systems. Their study is notoriously difficult and it is therefore particularly valuable to possess a zoo of tame, or reasonably tame, specimens. In this book, the author treats in careful detail the Mathieu functions, the Lamé functions, and spheroidal and ellipsoidal wave functions. The methods are as useful as the results, making the volume of interest to both the pure and applied mathematician.

Richard Bellman


This is a modest and informal book. By its size and its title it invites comparison with I. Prigogine's compact monograph, but the nature of each book is different. P. van Rysselberghe observes in his introduction that he "has avoided any discussion of fundamentals which would duplicate what is available in current textbooks. Having adopted as (his) main goal the rapid access to the practical handling of the machinery of "irreversible thermodynamics" he had to leave out of his presentation many theoretical fine points." The book will probably be of greatest interest to chemical engineers and others who need to relate the thermodynamics of irreversible processes to technological problems, though they may regret the lack of chapter-end examples and of an index.

P. D. Richardson


This book is based on a course in mathematical methods taught by the authors for some time at California Institute of Technology. It was, according to the authors, originally based on lectures from a course given by R. P. Feynman at Cornell University.

The contents follow the pattern: Ordinary Differential Equations; Infinite Series; Evaluation of Integrals; Integral Transforms; Complex Variables; Vectors and Matrices; Special Functions; Partial Differential Equations; Eigenfunctions, Eigenvalues and Green's Functions; Perturbation Theory; Integral Equations; Calculus of Variations; Numerical Methods; Probability and Statistics; Tensor Analysis and Differential Geometry; Group Theory.

The book appears to be well written, although perhaps too concisely. The references are extensive and useful comments accompany the references. The book depends heavily on examples which the reviewer feels adds to its attractiveness.

Although the form of the book reminds one somewhat of a guidebook, there is no question that it is a very useful one.

Rohn Truell


Just twenty years ago R. K. Luneburg gave a series of lectures on the Mathematical Theory of Optics at Brown University. The material appeared in mimeographed notes which were and, as far as the reviewer is aware, still are original and unique. The lectures were concerned with the methods of Hamilton and their application to geometrical optics as well as diffraction theory and, to a smaller

(Continued on p. 248)
extent, electron optics. The propagation of discontinuities together with the electromagnetic field equations and the method of characteristics is the basis of this work.

Making this material generally available is a worthwhile contribution in itself, and the lack of similar material in other texts makes the contribution more valuable.

Rohn Truell


The title of this book is somewhat deceptive because the book describes only a rather narrow class of stochastic processes. It is primarily an introduction to stochastic models of biological processes with emphasis on the practical aspects, the models themselves and methods of analysis, rather than the pure mathematical aspects of stochastic processes. Most of the book is devoted to Markov processes, particularly such things as the birth and death process, queueing, and epidemic models. Although most of this is for discrete state space processes, there is a valuable chapter on diffusion processes as limits of discrete ones. The prerequisites are an acquaintance with basic probability and statistics plus a typical undergraduate level familiarity with matrices and partial differential equations. Although the emphasis is on models of biological processes, the mathematics is the same as that which occurs in many other fields of application. This is certainly one of the best and one of the few books on methods and applications of the theory of Markov processes at an intermediate level. The book is somewhat repetitive but this may be an asset for the intended reader. This is a worthy addition to the Wiley Publications in Statistics which already contains many excellent books.

G. F. Newell


A simple branching process can be described roughly as follows: if the size of a population at time \( n \) is \( X_n \), the size \( X_{n+1} \) in the next generation will be the sum of \( X_n \) non-negative, independent, integer valued random variables—that is, each of the \( X_n \) elements gives birth homogeneously and independently to a random number of identical offspring. It is assumed that initially there is just one element, \( X_0 = 1 \). This simple scheme is modified and generalized in various ways to obtain more or less realistic models of many diverse biological and physical phenomena.

Originally developed to study the extinction of family names by Galton and Watson about 90 years ago, the subject was revived 15 or 20 years ago mainly under the stimulus of applications to genetics, cosmic rays, nuclear physics, etc. There now exists a large mass of papers both applied and theoretical on the subject, and to the reviewer's knowledge the present book is the first to give a systematic and comprehensive account of the accumulated research.

The main emphasis in this book is on a systematic mathematical development, but there are numerous illustrations and applications, sufficient so that the theorems never seem synthetic or contrived. The author has made an effort to incorporate the necessary probability and measure theoretic background in the text. This is minimal in the first part of the book, but later on when continuous time processes are considered greater demands are put on these prerequisites, without however impeding an understanding of the main results.

This book would appear indispensable to an applied mathematician whose area of research touches branching processes, and to the "pure" mathematician it offers a wealth of good analytical probability with many open problems of considerable importance.

D. A. Darling

(Continued on p. 256)

The study of random walk processes is of great interest to the mathematician because of the elegance and significance of the many results that can be obtained. It is of equal interest to the applied mathematician and mathematical physicist because of the variety of scientific processes that can be framed in these conceptually simple terms.

This book is devoted to random walk on the lattice points of domains in $N$-dimensional Euclidean space, with particular attention to the one- and two-dimensional cases. The connection with potential theory is discussed in clear and careful detail.


Apart from its worth as a compendium of recent research in this growing area, the volume is particularly valuable because of its lucidity of presentation, its detailed derivations and its painstaking organization. This is a book which was written to be read, and it is strongly recommended to those interested in pure and applied mathematics.

Richard Bellman

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This monograph has much to recommend it. At the same time it is also annoying in some respects. The material is substantially limited to the slow steady, laminar, incompressible isothermal flow of Newtonian fluids. It covers the equations of viscous flow, some exact solutions for rectilinear flow between parallel plates, Poiseuille flow, flow in wedge shaped regions, flow in pipes, flow past a sphere (Stokes and Oseen), some examples of rotary flow and an introduction to lubrication theory. About thirty-five pages are devoted to Cartesian and general tensors, not enough for one unacquainted with the subject to use as a text, and not necessary for those familiar with the topic. It is to be hoped that the day will soon come when it will not be thought necessary for every author to feel that he has to teach vector or tensor analysis in addition to his principal subject. Perhaps someday, someone in writing a monograph of this sort will have the courage to assume a knowledge of the fundamentals or to refer to texts on the subject where this ancillary material is treated in detail.

The subject matter proper of this book is treated in neat concise fashion. The style is lucid and readable. While there are not numerous references, enough are given to enable the reader who wants to dig deeper to find the necessary material.

The annoyances mentioned above are perhaps personal with this reviewer. He is not interested in the author's views on the Rumanian language, the streets of Providence, R. I. or any of the several other little gems which this reviewer feels the editor should have eliminated.

Finally, the indexing leaves something to be desired. For example no reference will be found to deformation, strain or strain-rate, topics which are treated. But "chemically reacting fluids," "magnetohydrodynamics," and "dissociating gas flow," will be found in the index. When these subjects are looked up, the reader will find in the text "We thereby exclude from our scope certain interesting and important areas of hydrodynamics—flow of chemically reacting fluids, magnetohydrodynamics, flow of dissociating gases . . . ."

Edward Saibel

(Continued on p. 264)

This book has the same flavor as the original Italian edition of 1956 except the material and bibliography have been updated to include the recent developments up to 1961. The book consists of nine chapters, all except three dealing with autonomous and nonautonomous second order systems. Second order systems are discussed in great detail and included are the following topics: the Poincaré-Bendixson theory, behavior near a singular point for general autonomous systems, self-excited oscillations in autonomous systems, bounded and periodic solutions of nonautonomous systems, and a brief discussion of structurally stable systems. There is an introductory chapter on existence, uniqueness, continuation and dependence of solutions on parameters for $n^{th}$ order systems. The last two chapters deal with general linear systems of $n^{th}$ order and stability and Lyapunov functions.

The book reads well, is relatively free of errors, and because of the detail on second order systems and the large bibliography, it is recommended as a very good reference.

Jack K. Hale


This book is the proceedings of a conference held at the University of California, Los Angeles, January 30–31, 1964. The rapidity of publication is commendable. The particular variational optimization problem investigated by the participants at this conference involves a dynamical system described by deterministic differential equations containing unspecified control functions that are to be determined. Neither stochastic nor adaptive variational problems, nor linear programming problems, nor any other type of optimization problem is considered.


The paper by Hestenes outlines a method of establishing necessary conditions for the solution of the control problem and touches upon the relationship between the Pontryagin maximum principle and classical methods and results. The Storey-Rosenbrock contribution treats one problem by several quite different computing methods. Since most current methods involve as much art as science, the resulting evaluation of methods is not necessarily valid, but the idea is a good one and more studies of this sort would be welcome. Most of the remaining papers explain and illustrate particular methods for treating particular classes of problems.

S. E. Dreyfus

(Continued on p. 288)

This book, intended for an undergraduate text in Classical Dynamics, is based on the author's conviction that the traditional ways of teaching the subject must be abandoned in favour of a more systematic approach, characterized by close attention to the precise meaning of each concept or postulate. Probably the most conspicuous consequence of this attitude consists in the fact that (in contrast to the concepts of space and time) the concept of force is not introduced, as in most texts, without definition, at the outset, but gradually developed out of a treatment in which the concept of "additional acceleration" plays the key-role.

The first chapter is devoted to the Gravitational Theory of Planetary Systems. It contains some remarkable sections on the n-body problem, its virial theorems and the dispersion of the system, also on gravitation between finite bodies and on the figure of the Earth. The second chapter starts with reference frames and relative motion and develops the general principles of mechanics. A few sections on Continuum Mechanics precede the third chapter which is devoted to rigid bodies, in particular gyroscopes.

One will observe that the treatment is unusual. The reviewer doubts whether this is really an introductory text. The concept of "additional acceleration," introduced without any previous explanation of rigid-body-kinematics hardly seems more transparent than the force concept. From a more advanced point of view, however, the book deserves to be warmly recommended. The approach is original; the philosophy underlying it is extremely critical, and the book presents a wealth of information usually not found in similar texts.

H. Ziegler


The term "Flächenschluss" is used to indicate the arrangement of congruent figures to fill a plane region without gaps or overlaps. Although the engineering problem of cutting a sheet into congruent parts without unnecessary waste provided the motive for the book, a considerable number of pages are devoted to the necessary mathematical background, which is not part of the traditional engineering curriculum in mathematics. After a brief introductory chapter, the seventeen discrete groups of isometries are discussed in Chapter 2. The next chapter applies these results to the problem on hand. In Chapter 4, the authors classify the fundamental regions according to the manner in which their boundaries are built up from pairs of congruent arcs, obtaining twenty eight basic types, which are described in considerable detail. In Chapter 5, these results are specialized to polygonal regions. In Chapter 6, fundamental regions are discussed whose boundaries contain, among others, several arcs of a given circle. The last chapter is concerned with manufacturing problems.

W. Prager


The first part of the book starts with a brief introduction to FORTRAN II. There follows a brief survey of the principal methods of numerical analysis including some difference methods for the approximate solution of partial differential equations. In the second part, these methods are illustrated by complete programs in FORTRAN II. The study of these programs is facilitated by flow charts, various hints for the beginner, small numerical examples, and numerous exercises. The programs are suited for small machines, such as the IBM 1620.

The introduction to FORTRAN as well as the presentation of numerical methods are rather concise. As a consequence of this, the beginner will frequently have to consult additional works. As collateral reading in connection with a course on programming and numerical methods, the book will however prove useful.

B. Meister