# ON ASYMPTOTIC STABILITY OF NONLINEAR HEREDITARY PHENOMENA*

BY

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**Summary.** Conditions for long-term stability of nonlinear hereditary phenomena are derived from two Tauberian theorems. When the hereditary phenomena are time invariable (i.e., of the closed cycle type), then the asymptotic limits of the phenomena are evaluated. An application is made to the investigation of the stability of a rigid column restrained by a nonlinear viscoelastic element.

**1. Introduction.** In many problems encountered in linear hereditary phenomena, the governing equations defining the response, $b(t)$, of physical systems subjected to some prescribed input, $g(t)$, are linear integral equations of the form

$$\int_{-\infty}^{t} b(\tau)f(t;\tau)\,d\tau = g(t) \tag{1}$$

where the generalized function $f(t;\tau)$ is the hereditary influence function of the system.

When interest is centered on problems of the stability of the system, then it is generally desired to determine conditions on $f(t;\tau)$ (given some specified properties of $g(t)$) under which $b(t)$ is bounded. Provided the "imperfect" kernel (time variable kernel) possess suitable asymptotic properties, the boundedness of the output function $b(t)$ follows from a very general Tauberian theorem due to Pitt [1]. In many cases of interest the system is time invariable, so that the kernel $f(t;\tau)$ in equation (1) is of the type $f(t-\tau)$. For this kind of kernel, and provided $g(t)$ have a finite asymptotic limit, another, somewhat less general, Tauberian theorem due to Paley and Wiener [2] allows, not only for the determination of the conditions under which the system is stable, but also for the evaluation of the asymptotic value of $b(t)$. Applications of these theorems to the study of the stability of a linear viscoelastic column were made in references [3] and [4].

In this paper it is shown how these theorems may be extended to the investigation of the stability of nonlinear hereditary phenomena, represented by general nonlinear functionals. As an application, the procedures developed are applied to the analysis of the stability of a rigid bar, stabilized by a nonlinear "viscoelastic spring."

**2. Inversion and asymptotic behavior of the nonlinear functional.** Given a mapping between two functions $y(t)$ and $x(t)$

$$y(t) = \mathfrak{F}[\underset{\tau=-\infty}{\overset{\tau=t}{x(\tau)}}] \tag{2}$$

where $\mathfrak{F}$ is a nonlinear functional, then if $\mathfrak{F}$ is suitably well-behaved, (2) can be represented in terms of a Volterra–Fréchet expansion [5]

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$$y(t) = \int_{-\infty}^{t} x(\tau)K_1(t;\tau)\,d\tau + \frac{1}{2!}\int_{-\infty}^{t}\int_{-\infty}^{t} x(\tau_1)x(\tau_2)K_2(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2 + \cdots \tag{3}$$

where the kernel functions $K_n(t;\tau_1,\tau_2,\cdots,\tau_n)$ (which in general are distributions) are identically zero if any of the arguments $\tau_i$ are greater than the argument $t$. It is noted that, without any loss of generality, the kernel functions $K_n$ can be taken to be symmetric in the $n$ arguments $\tau_1,\tau_2,\cdots,\tau_n$.

A problem of great interest in applications is the determination of the conditions for which the asymptotic boundedness of the function $x(t)$ is assured, assuming that the function $y(t)$ is bounded as $t \to \infty$. In order to carry out this investigation, it is convenient to invert equation (3) so as to obtain $x(t)$ as an explicit functional of $y(t)$. Equation (3) can be inverted by using the algorithm established by Volterra [5], [6], yielding

$$x(t) = F_1(t) + \frac{1}{2!}F_2(t) + \frac{1}{3!}F_3(t) + \cdots \tag{4}$$

where the functions $F_n(t)$ are obtained from the inversion of the following triangular system of linear integral equations

$$\begin{aligned}
&\int_{-\infty}^{t} F_1(\tau)K_1(t;\tau)\,d\tau = y(t),\\
&\int_{-\infty}^{t} F_2(\tau)K_1(t;\tau)\,d\tau = -\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)K_2(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2,\\
&\int_{-\infty}^{t} F_3(\tau)K_1(t;\tau)\,d\tau = -\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)F_1(\tau_3)K_3(t;\tau_1,\tau_2,\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3\\
&\qquad\qquad - 3!\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)K_2(t;\tau_1,\tau_2)\,d\tau_1\,d\tau_2,\\
&\qquad\qquad \cdots
\end{aligned} \tag{5}$$

It is recognized that the inversion of this system, and consequently of (3), depends upon the existence of the kernel reciprocal to $K_1(t;\tau)$ which allows for the resolution of the system. It may be noted that the restrictions on $K_1(t;\tau)$ for the existence of the resolvent kernel are quite weak [7]. However, in the investigation of the asymptotic behavior of equations of the type (3) and (4), slightly stronger restrictions on the kernels have to be imposed.

From (4) it may be seen that the condition for asymptotic boundedness of $x(t)$ depends upon the conditions which ensure asymptotic boundedness of all the functions $F_n(t)$ given by equations (5). A very convenient and general theorem which can be used to establish the conditions ensuring asymptotic boundedness of the functions $F_n(t)$ is a Tauberian theorem developed by Pitt [1]. Application of the theorem to the first of (5) yields that if a function $\bar{K}_1(t-\tau)$ can be constructed so that it approximates $K_1(t;\tau)$ in the sense that

$$\lim_{t\to\infty}\int_{-\infty}^{t} |K_1(t;\tau) - \bar{K}_1(t-\tau)|\,e^{-\eta(t-\tau)}\,d\tau = 0 \tag{6}$$

for some $\eta > 0$, and if

$$\int_{-\infty}^{\infty} \bar{K}_1(\tau)e^{-\nu\tau}\, d\tau \neq 0, \qquad \operatorname{Re} \nu > 0 \tag{7}$$

then $F_1(t)$ is bounded as $t \to \infty$.

The conditions for asymptotic boundedness of the functions $F_2(t)$, $F_3(t)$, $\cdots$ are the same as was required for the function $F_1(t)$, given by equations (6) and (7), with the additional requirement that

$$\int_{-\infty}^{t} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} K_n(t; \tau_1, \tau_2, \cdots, \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n < \infty \tag{8}$$

for all $t$.

If (3) is used to represent the constitutive law of a general nonlinear, aging viscoelastic material with $y(t)$ representing the strain and $x(t)$ the stress, then condition (8) is equivalent to restricting attention to materials with bounded creep.

In the investigation of stability of physical systems, (6) and (7) are generally the pertinent relations which determine the conditions ensuring stability. When the physical system is time invariant, then (6) is identically satisfied and only (7) remains as the pertinent condition for stability.

Although the previous analysis leads to conditions for asymptotic boundedness under relatively weak restrictions—so that the results will be useful in applications—it does not give an indication of a value of a lower or upper asymptotic bound for the function $x(t)$. The previous analysis did not require the existence of an asymptotic limit for $y(t)$. If, in fact, an asymptotic limit of $y(t)$ exists, then for a certain class of functionals it is possible to evaluate the asymptotic limit of $x(t)$, even with a very limited knowledge of the kernel functions. This is the case of functionals of the closed cycle type (which are related to time invariant physical systems) where only the asymptotic values of the integrals of the kernels are required. This is treated in detail in the following section.

**3. Asymptotic limits of closed cycle type functionals.** In what follows, only functionals of the closed cycle type (in the sense of Volterra) will be considered. In that case, (3) becomes

$$y(t) = \int_{-\infty}^{t} x(\tau)K_1(t - \tau)\, d\tau + \frac{1}{2!}\int_{-\infty}^{t}\int_{-\infty}^{t} x(\tau_1)x(\tau_2)K_2(t - \tau_1, t - \tau_2)\, d\tau_1\, d\tau_2 + \cdots \tag{9}$$

where the functions $K_n$ are identically zero if any of their arguments are negative.

For such a functional, two different problems of interest may be distinguished. The first problem, which is a quite straightforward one, is the determination of the asymptotic limit of the function $y(t)$ when $x(t)$ is assumed to have an asymptotic limit. The asymptotic limit of $y(t)$, in terms of $x(t)$ and the kernels $K_n$, is given by

$$\lim_{t\to\infty} y(t) = y(\infty) = x(\infty)\int_{-\infty}^{\infty} K_1(\tau)\, d\tau + \frac{1}{2!}x^2(\infty)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_2(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 + \cdots \tag{10}$$

where $x(\infty) = \lim_{t\to\infty} x(t)$ provided

$$I_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K_n(\tau_1, \tau_2, \cdots, \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n < \infty. \tag{11}$$

The expression (10) indicates that the asymptotic value of $y(t)$ does not depend on the past history of $x(t)$, but only on its asymptotic value $x(\infty)$. It may be noted that if the principle of dissipation of hereditary action (as stated by Volterra [5], [8]) were to be assumed, then the result obtained from (10) would follow as an immediate consequence. In this paper, however, the principle of dissipation of hereditary action follows from (10).

The other problem of interest is the determination of the actual value of the asymptotic limit of $x(t)$ when the value of the asymptotic limit of $y(t)$ is known. This limit may be evaluated from (4) and (5), where it should be kept in mind that in this discussion the kernel functions $K_n$ are time invariant so that equations (5) now read

$$\begin{aligned}
&\int_{-\infty}^{t} F_1(\tau)K_1(t-\tau)\, d\tau = y(t),\\
&\int_{-\infty}^{t} F_2(\tau)K_1(t-\tau)\, d\tau = -\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)K_2(t-\tau_1, t-\tau_2)\, d\tau_1\, d\tau_2,\\
&\int_{-\infty}^{t} F_3(\tau)K_1(t-\tau)\, d\tau\\
&\quad = -\int_{-\infty}^{t}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)F_1(\tau_3)K_3(t-\tau_1, t-\tau_2, t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3\\
&\qquad - 3!\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_2(\tau_2)K_2(t-\tau_1, t-\tau_2)\, d\tau_1\, d\tau_2\\
&\cdots
\end{aligned} \tag{12}$$

By means of the theorem of Paley and Wiener [2], already mentioned in the introduction, the limit of the functions $F_n(t)$ in (12) may be obtained as follows:

$$\begin{aligned}
\lim_{t\to\infty} F_1(t) &= F_{1\infty} = \lim_{t\to\infty} y(t) \Big/ \int_{-\infty}^{\infty} K_1(\tau)\, d\tau = y(\infty)/I_1,\\
\lim_{t\to\infty} F_2(t) &= F_{2\infty} = -\lim_{t\to\infty}\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)K_2(t-\tau_1, t-\tau_2)\, d\tau_1\, d\tau_2 \Big/ \int_{-\infty}^{\infty} K_1(\tau)\, d\tau\\
&= -F_{1\infty}^2 I_2/I_1,\\
\lim_{t\to\infty} F_3(t) &= F_{3\infty} = -\lim_{t\to\infty}\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_1(\tau_2)F_1(\tau_3)\\
&\qquad \cdot K_3(t-\tau_1, t-\tau_2, t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 \Big/ \int_{-\infty}^{\infty} K_1(\tau)\, d\tau\\
&\qquad - \lim_{t\to\infty} 3!\int_{-\infty}^{t}\int_{-\infty}^{t} F_1(\tau_1)F_2(\tau_2)K_2(t-\tau_1, t-\tau_2)\, d\tau_1\, d\tau_2 \Big/ \int_{-\infty}^{\infty} K_1(\tau)\, d\tau\\
&= -\{F_{1\infty}^3 I_3 + 3!\, F_{1\infty}F_{2\infty}I_2\}/I_1,\\
&\cdots
\end{aligned} \tag{13}$$

provided condition (11) holds and

$$\int_{-\infty}^{\infty} K_1(\tau)e^{-\nu\tau}\, d\tau \neq 0, \qquad \operatorname{Re} \nu > 0 \tag{14}$$

Then it follows that the asymptotic value of the function $x(t)$ derived from (4) is

$$\lim_{t\to\infty} x(t) = x(\infty) = \sum_{n=1}^{\infty} \frac{1}{n!} F_{n\infty}, \tag{15}$$

where $F_{n\infty}$ are given by equations (13). Equation (15) is nothing but the formal algebraic inversion of equation (10). It is of importance to point out that beside the restriction (11) which involves all of the kernels, the further, important restriction (14) is imposed solely on the first order kernel $K_1(t)$.

Once again it is desired to emphasize that equations (7) or (14) are generally the pertinent basic relations for the determination of the conditions of long-time (or asymptotic) stability of physical systems of the hereditary type. In the next section, an example is presented to illustrate this point.

**4. Stability of a viscoelastically restrained rigid bar.** Consider a viscoelastically restrained rigid bar submitted to constant axial and lateral forces $P$ and $F$, respectively, as shown in Fig. 1, where the restraint element $AB$ is assumed to be made from a nonaging, nonlinear viscoelastic material. At any instant, equilibrium of the bar requires that the force $H(t)$ in the viscoelastic rod $AB$ be given by

$$H(t) = k\omega(t) + F; \qquad k = \frac{P}{h} \tag{16}$$

where $\omega(t)$ is the lateral deflection of the tip of the bar, assumed to be small compared to $h$.

The force-displacement relationship for the viscoelastic restraining rod is assumed to be given by a general nonlinear functional

$$\omega(t) = \mathfrak{F}\Big[H(\tau)\Big]_{\tau=-\infty}^{\tau=t} \tag{17}$$

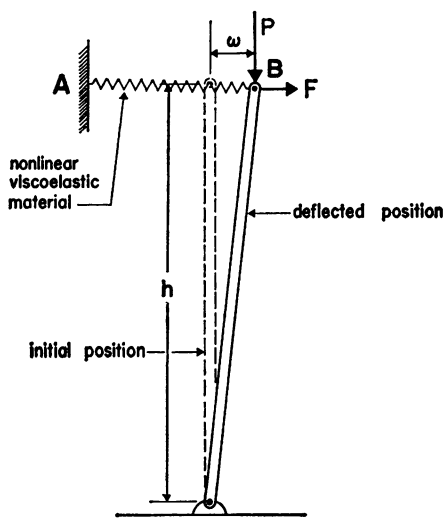


FIG. 1.

where the system is assumed to be completely quiescent for $t < t_0$ and all loads are applied at $t = t_0$. Utilizing the Volterra–Fréchet expansion to represent the functional, (17) may be written as

$$\omega(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t} \int_{t_0}^{t} \cdots \int_{t_0}^{t} H(\tau_1)H(\tau_2) \cdots H(\tau_n) \cdot f_n(t - \tau_1, t - \tau_2, \cdots, t - \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{18}$$

where the generalized functions $f_n$ are the material kernels. Here, the creep of the material is assumed to be bounded so that

$$\lim_{t\to\infty} \epsilon^{(n)}(t - t_1, t - t_2, \cdots, t - t_n) < \infty \tag{19}$$

for any values of $t_i$ where

$$\epsilon^{(n)}(t - t_1, t - t_2, \cdots, t - t_n) = \int_{t_1}^{t} \int_{t_1}^{t} \cdots \int_{t_n}^{t} f_n(t - \tau_1, t - \tau_2, \cdots, t - \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{20}$$

Substituting $H(t)$ from (16) into (18) and carrying out the corresponding expansions, the following nonlinear integral equation for the displacement $\omega(t)$ is obtained

$$\omega(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} k^i F^{n-i} \int_{t_0}^{t} \int_{t_0}^{t} \cdots \int_{t_0}^{t} \omega(\tau_1)\omega(\tau_2) \cdots \cdot \omega(\tau_i) f_n(t - \tau_1, t - \tau_2, \cdots, t - \tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{21}$$

Grouping terms of equal order, the following equation results

$$\begin{aligned} -G_0(t - t_0) &= \int_{t_0}^{t} \omega(\tau_1)[G_1(t - \tau_1, t - t_0) - \delta(t - \tau_1)]\, d\tau_1 \\ &+ \frac{1}{2!} \int_{t_0}^{t} \int_{t_0}^{t} \omega(\tau_1)\omega(\tau_2) G_2(t - \tau_1, t - \tau_2, t - t_0)\, d\tau_1\, d\tau_2 \\ &+ \frac{1}{3!} \int_{t_0}^{t} \int_{t_0}^{t} \int_{t_0}^{t} \omega(\tau_1)\omega(\tau_2)\omega(\tau_3) G_3(t - \tau_1, t - \tau_2, t - \tau_3, t - t_0)\, d\tau_1\, d\tau_2\, d\tau_3 + \cdots \end{aligned} \tag{22}$$

where $\delta(t)$ is the Dirac delta function and

$$G_r(t - \tau_1, t - \tau_2, \cdots, t - \tau_r, t - t_0) = \sum_{n=1}^{\infty} \frac{r!}{n!} \binom{n}{r} k^r F^{n-r} \int_{t_0}^{t} \int_{t_0}^{t} \cdots \int_{t_0}^{t} f_n(t - \tau_1, t - \tau_2, \cdots, t - \tau_n)\, d\tau_{r+1}\, d\tau_{r+2} \cdots d\tau_n \tag{23}$$

To investigate the conditions for asymptotic boundedness of the deflection $\omega(t)$ in (22), the results obtained in sections II and III may be applied. In view of equation (19) and (20), the infinite integral of the kernels $G_r$ satisfies the boundedness requirements expressed by (8). The kernel $K_1(t; \tau)$ appearing in equation (6) is given in the problem under consideration by

$$K_1(t;\tau) = G_1(t-\tau, t-t_0) - \delta(t-\tau)$$
$$= k\sum_{n=1}^{\infty}\frac{1}{(n-1)!}F^{n-1}\left[-\frac{\partial}{\partial\tau}\epsilon^{(n)}(t-\tau, t-t_0, \cdots, t-t_0)\right] - \delta(t-\tau). \tag{24}$$

A kernel $\bar{K}_1(t-\tau)$ which approximates $K_1(t;\tau)$ in the sense of equation (6) is given by

$$\bar{K}_1(t-\tau) = G_{1\infty}(t-\tau) - \delta(t-\tau) \tag{25}$$

where

$$G_{1\infty}(t-\tau) = G_1(t-\tau, \infty)$$
$$= k\sum_{n=1}^{\infty}\frac{1}{(n-1)!}F^{(n-1)}\left[-\frac{\partial}{\partial\tau}\epsilon^{(n)}(t-\tau, \infty, \cdots, \infty)\right] \tag{26}$$

and where it is noted that the infinite integral of $G_{1\infty}$ is bounded on account of (19) and (20).

Substituting the value of $\bar{K}_1$ given by (25) into (7), the following condition for asymptotic boundedness of the deflection $\omega(t)$ is obtained

$$\int_{-\infty}^{\infty}\bar{K}_1(\tau)e^{-\nu\tau}\,d\tau$$
$$= k\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{1}{(n-1)!}F^{n-1}\frac{\partial}{\partial\tau}\epsilon^{(n)}(\tau, \infty, \cdots, \infty)\right\}e^{-\nu\tau}\,d\tau - 1 \neq 0, \qquad \operatorname{Re}\nu > 0 \tag{27}$$

Since creep of the viscoelastic restraint is assumed to increase with time and with the load on the restraint, the expression within the braces in equation (27) will be positive; consequently, the above equation is satisfied if and only if

$$k\sum_{n=1}^{\infty}\frac{1}{(n-1)!}F^{n-1}\epsilon^{(n)}(\infty, \infty, \cdots, \infty) < 1. \tag{28}$$

This equation provides a sufficient condition for asymptotic boundedness of the deflection $\omega(t)$. (It is also a necessary condition provided some further weak restrictions on the kernels $\epsilon^{(n)}$ are assumed.) Thus (recalling the definition of $k$) the critical load, $P^*$, for asymptotic boundedness of the deflection is given by

$$P^* = h\Big/\sum_{n=1}^{\infty}\frac{1}{(n-1)!}F^{n-1}\epsilon^{(n)}(\infty, \infty, \cdots, \infty) \tag{29}$$

obtained by substituting an equality sign for the inequality sign in (28).

It is important to note (as clearly indicated by (29)) that the critical load does not depend on a complete knowledge of the material kernels appearing in the constitutive equation (18), but only on their asymptotic values.

A useful physical interpretation of the asymptotic critical load given by (29) follows from comparison of this equation with the expression for the instantaneous critical load. The equation governing the instantaneous deflection of the rod may be obtained by setting $t = t_0^+$ in the general result expressed by (22), making use of the definition given by (20). This yields the following equation

$$-C_0 = \omega(C_1 - 1) + \omega^2\frac{C_2}{2!} + \omega^3\frac{C_3}{3!} + \cdots \tag{30}$$

where

$$C_r = \sum_{n=1}^{\infty} \frac{r!}{n!} \binom{n}{r} k^r F^{n-r} \epsilon^{(n)}(0^+, 0^+, \cdots, 0^+), \tag{31}$$

follows from equation (23), and where, as may be seen from equation (18),

$$\epsilon^{(n)}(0^+, 0^+, \cdots, 0^+)$$

are coefficients in the power series expansion for the instantaneous elastic response of the material of the restraint at $t = t_0^+$.

Equation (30) may be solved for $\omega$ as an explicit function of the $C_r$ (and, therefore, of the axial load parameter $k$) by using Volterra's algorithm previously described in Section II. It is noted, of course, that the algorithm serves to invert nonlinear algebraic equations of the form of equation (30) as well as nonlinear integral equations of the form of equation (3), since equation (30) is just a special case of equation (3). From the inverted form of equation (30), it follows immediately that if $C_1 \to 1$ then $\omega \to \infty$, so that the critical condition is

$$C_1 = 1. \tag{32}$$

This represents the condition for instantaneous instability. Utilizing equation (31) and the definition of $k$, condition (32) yields for the instaneous critical load $P^I$ the expression

$$P^I = h \Big/ \sum_{n=1}^{\infty} \frac{1}{(n-1)!} F^{n-1} \epsilon^{(n)}(0^+, 0^+, \cdots, 0^+). \tag{33}$$

It is noted that if the instantaneous elastic deformation of the restraint is assumed to be a strictly increasing function of the force on the restraint, then the denominator in equation (33) will be positive.

The above equation, compared with equation (29), shows that the asymptotic critical load can be regarded as the critical load of the bar having a nonlinear elastic restraint with a nonlinear elastic force-displacement ($\omega$, $H$) relationship given by

$$\omega = \sum_{n=1}^{\infty} \frac{1}{n!} H^n \epsilon^{(n)}(\infty, \infty, \cdots, \infty) \tag{34}$$

which is identical to the asymptotic limit of equation (18) under a constant load $H$.

It is of interest to note (as may be seen from equations (18) and (20)) that the denominators of equations (29) and (33) are, respectively, the derivatives (with respect to $H$) at $H = F$ of the infinity and zero isochronous curves [9] of the restraint. These quantities may be readily obtained from a sequence of creep tests performed on the nonlinear viscoelastic restraint around $H = F$.

When the perturbing force, $F$, approaches zero, the asymptotic and instantaneous critical loads given by equations (29) and (33) reduce to

$$P^* = h/\epsilon^{(1)}(\infty), \tag{35}$$

$$P^I = h/\epsilon^{(1)}(0^+), \tag{36}$$

establishing the fact that in this case both critical loads are governed solely by the linear kernel, as might be expected.

The problem treated in this paper is a very particular one, specifically simplified so as to illustrate, without unnecessary complications (but, however, without great loss of generality), an application of the previously developed theory. A solution of a more complex and general problem concerning creep buckling is forthcoming.

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