BIFURCATION OF PERIODIC SOLUTIONS IN A NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATION OF NEUTRAL TYPE*

BY

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Abstract. The existence of a self-sustained periodic solution in the autonomous equation

$$u'(\tau) - \alpha u'(\tau - h) + \beta u(\tau) + \alpha\gamma u(\tau - h) = \epsilon f(u(\tau))$$

is proved under appropriate assumptions on α, β, γ, f and h. The method of proof consists in converting this equation into an equivalent nonlinear integral equation and demonstrating the convergence of an appropriate iteration scheme.

In this paper we consider the equation

$$u'(\tau) - \alpha u'(\tau - h) + \beta u(\tau) + \alpha\gamma u(\tau - h) = \epsilon f(u(\tau)), \tag{1}$$

where $h \geq 0$ and $\alpha = \alpha_0(1 + \epsilon)$. The existence of a periodic solution will be proved for small $\epsilon \geq 0$ under appropriate assumptions on the parameters α_0, β, γ, and f. The left-hand side of this equation is a linear difference-differential operator of neutral type (for a definition see [1]). The existence of periodic solutions for functional-differential equations which include difference-differential equations of retarded type but not neutral type has been discussed by Krasovskii [2], Shimanov [3], [4], and Hale [5]. In all these cases, the equations are of the forced type where the right-hand side is a 2π-periodic function of τ. Equation (1), which we consider, is autonomous, and we look for a self-sustained oscillation.

Difference-differential equations of the type (1) arise from electrical networks such as the one shown in Fig. 1. The equations of this network are

$$\begin{aligned} L(\partial i/\partial t) &= -\partial v/\partial x, \\ C(\partial v/\partial t) &= -\partial i/\partial x, \end{aligned} \qquad 0 \leq x \leq 1 \tag{2}$$

$$E - v(0, t) - Ri(0, t) = 0, \qquad C_1(dv(1, t)/dt) = i(1, t) - g(v(1, t)),$$

where L, C are the specific inductance and capacitance in the transmission line. The question of the existence of a periodic solution of some unknown period T can be posed by giving the additional boundary conditions

$$v(x, 0) = v(x, T), \qquad i(x, 0) = i(x, T).$$

Thus, we have a boundary-value problem for a hyperbolic partial differential equation with boundary conditions given on the rectangle shown in Fig. 2. Of course, in general, a boundary-value problem for a hyperbolic equation is not well posed. The difference here is that the boundary $t = T$ is free to be chosen.

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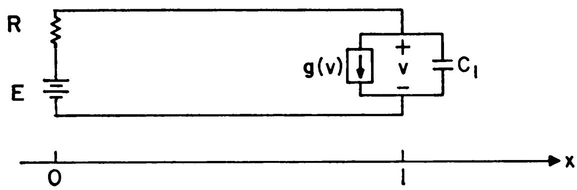

FIG. 1. Transmission Line Network

As described in [6], Eqs. (2) can be written in the form

$$C(v'(t) + Kv'(t - h)) + (1/z - g^*)v(t) - K(1/z + g^*)v(t - h) = f(v(t), v(t - h)), \quad (3)$$

where $K = (R - z)/(R + z)$, $z = (L/C)^{1/2}$, $h = 2(LC)^{1/2}$, $f(v, w) = O(v^3 + w^3)$ and g^* is the derivative of $g(v)$ at some point $v = v_0$. Clearly, Eq. (3) can be brought into the form of Eq. (1) with the transformation $v = \epsilon^{1/2}u$ except for the fact that f depends on $v(t - h)$. However, in what follows, there is no essential difficulty in handling this case, and for simplicity we neglect this generalization.

In Eq. (1), let $t = \omega\tau$ where ω is unknown, and let us seek a periodic solution of period 2π in t. With $v(t) = u(\tau)$ Eq. (1) becomes

$$L_\epsilon[v] \equiv \omega(v'(t) - \alpha v'(t - \omega h)) + \beta v(t) + \alpha\gamma v(t - \omega h) = \epsilon f(v(t)). \quad (4)$$

The linear operator L_ϵ is a difference-differential operator of neutral type and, as we shall see, the parameter α_0 appearing in $\alpha = \alpha_0(1 + \epsilon)$ will be chosen so that there exists exactly one pair of purely imaginary characteristic roots of L_0.

The characteristic equation associated with the operator L_ϵ is

$$q(s\omega) \equiv s\omega(1 - \alpha \exp(-s\omega h)) + \beta + \alpha\gamma \exp(-s\omega h) = 0, \quad (5)$$

where s is a complex number. We shall prove the existence of periodic solutions of (4) for ϵ small under the condition that for $\alpha = \alpha_0$ Eq. (5) has roots $s\omega = \pm i\omega_0$. First, we prove the following

LEMMA 1. For $\gamma > \beta > 0$ there exists an infinite set of real pairs (α_0, ω_0) such that

$$i\omega_0(1 - \alpha_0 \exp(-i\omega_0 h)) + \beta + \alpha_0\gamma \exp(-i\omega_0 h) = 0, \quad (5a)$$

where $\alpha_0^2 < 1$.

PROOF: By writing (5a) in its real and imaginary parts and solving for $\sin \omega_0 h$,

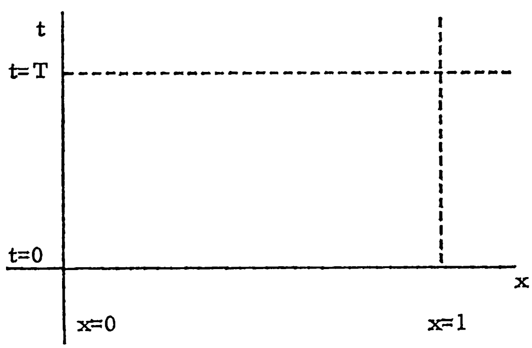

FIG. 2. Periodic Cell

$\cos \omega_0 h$, we find

$$\sin \omega_0 h = \frac{\omega_0}{\alpha_0} \frac{\gamma + \beta}{\omega_0^2 + \gamma^2}, \tag{6}$$

$$\cos \omega_0 h = \frac{1}{\alpha_0} \frac{\omega_0^2 + \gamma\beta}{\omega_0^2 + \gamma^2}, \tag{7}$$

which can be combined to yield

$$\omega_0^2 = \frac{\alpha_0^2\gamma^2 - \beta^2}{1 - \alpha_0^2}, \tag{8}$$

$$\tan \omega_0 h = \frac{\omega_0(\gamma + \beta)}{\omega_0^2 - \gamma\beta}. \tag{9}$$

The solutions of Eq. (9), which is independent of α_0, can be found graphically as shown in Fig. 3. Clearly for any ω_0 satisfying (9), there exists an α_0^2 such that (8) holds because $(\alpha_0^2\gamma^2 - \beta^2)/(1 - \alpha_0^2)$ can be made to range between $-\beta^2$ and $+\infty$ by varying α_0^2 between 0 and 1. Eqs. (8) and (9) give pairs ω_0, α_0^2 and by taking the correct sign on α_0, the pairs ω_0, α_0 satisfy (5a). Arranging the ω_0 in increasing order starting with $\omega_0 = 0$ the corresponding α_0 alternate in sign starting with $\alpha_0 = -\beta/\gamma$.

Note that for $\alpha = 0$, there is only one root of (5), namely $s\omega = -\beta < 0$. As α^2 is increased, there is an infinity of roots which all lie in the left-half $s\omega$ plane. When $\alpha = -\beta/\gamma$, we have the first root crossing the imaginary axis at $s\omega = 0$. Similarly, when α equals the first positive α_0, we have a pair of roots $s\omega = \pm i\omega_0$ crossing the imaginary axis. The remainder of the roots of (5) lie in the left-half plane because of continuity with respect to α. For all other pairs α_0, ω_0 there is at least one root in the right half-plane. For this reason we expect that the periodic solution associated with the pair α_0, ω_0 corresponding to the first positive value of α_0 is the only stable periodic solution.

However, since existence and stability are independent concepts, the following theorem is applicable for any solution α_0, ω_0 of (5a). The theorem states the existence of a periodic solution* near the function $v = 2a^* \cos t$ where a^* satisfies the bifurcation equation

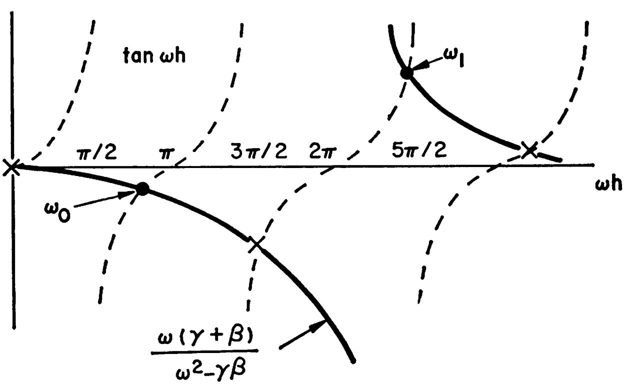

FIG. 3. Graphical Solution of Equation (9)

*By solution, we mean a continuously differentiable function $v(t)$ which satisfies (4) point-wise.

$$a^* = \frac{1}{2\pi x_0} \int_0^{2\pi} f(2a^* \cos t) \cos t \, dt \tag{10}$$

and x_0 is specified by (12) below. If $a^* = 0$, the theorem is trivial, and therefore, we rule this out. An example of a nonlinear function f where there is a nonzero solution of (10) is $f(v) = -v^3$. Then the solution is a $a^* = [-x_0/3]^{1/2}$ which is real since it is shown below that $x_0 < 0$.

Theorem. Let (α_0, ω_0) be any pair satisfying (5a) where $\omega_0 \neq 0$ and assume that there exists a real solution $a^* \neq 0$ of

$$F(a^*) \equiv a^* - \frac{1}{2\pi x_0} \int_0^{2\pi} f(2a^* \cos t) \cos t \, dt = 0$$

such that $|\partial F/\partial a|_{a=a^*} \geq \mu > 0$ where x_0 is specified by (12) below. Also assume that f is continuously differentiable, $f(0) = f'(0) = 0$, and $|f'(v)| < \nu$ for $|v| < 4\,|a^*|$. Then for small $\epsilon \geq 0$, there exist functions $\omega = \omega(\epsilon)$, $v = v(t, \epsilon)$ such that $v(t, \epsilon)$ is a 2π-periodic solution of (4) with $\omega = \omega(\epsilon)$. Furthermore, $\omega(0) = \omega_0$ and $v(t, 0) = 2a^* \cos t$.

Note. For $\epsilon = 0$, a two-parameter family of periodic solutions exists of the form $v = ae^{it} + \bar{a}e^{-it}$ where a is an arbitrary complex number. This follows since (5a) implies that $L_0[e^{\pm it}] = 0$. However, since the equation is autonomous, this family is a one-parameter family modulo an arbitrary translation of time. We represent this family by $v(t) = 2a \cos t$ where a is an arbitrary real number. The above theorem implies that as ϵ becomes positive the solution $v(t) = 2a^* \cos t$ bifurcates into a neighboring periodic solution with period in the τ variable equal to $2\pi/\omega(\epsilon)$.

Proof of Theorem: The first part of the proof will be to transform Eq. (4) into an integral equation. For this purpose we make the following definition

$$q_\epsilon(ik) \equiv \exp(-ikt) L_\epsilon[\exp(ikt)]$$

and establish some properties of $q_\epsilon(ik)$. Clearly from (4) or (5),

$$q_\epsilon(ik) = ik\omega(1 - \alpha \exp(-ik\omega h)) + \beta + \alpha\gamma \exp(-ik\omega h), \tag{11}$$

where $\alpha = \alpha_0(1 + \epsilon)$ and $\omega = \omega(\epsilon)$ is an unspecified function of ϵ. If $(\alpha_0, \omega(0))$ is a pair satisfying (5a), then

$$q_0(\pm i) = 0.$$

Let $\omega = \omega_0 + \epsilon\omega_1$ and define $x(\omega_1, \epsilon) + iy(\omega_1, \epsilon) \equiv q_\epsilon(i)/\epsilon$. Now x is bounded away from zero if ϵ is small enough and $\omega_1 \geq 0$. This follows since

$$x_0 \equiv \lim_{\epsilon \to 0} x(\omega_1, \epsilon) = \omega_1(-\alpha_0 \sin \omega_0 h - \alpha_0\omega_0 h \cos \omega_0 h - \alpha_0 \gamma h \sin \omega_0 h) + \alpha_0\gamma \cos \omega_0 h - \alpha_0\omega_0 \sin \omega_0 h.$$

Using (6) and (7) this becomes

$$x_0 = \omega_1 \left[\frac{-\omega_0(\gamma + \beta + h(\omega_0^2 + \gamma^2))}{\omega_0^2 + \gamma^2} \right] - \beta. \tag{12}$$

Hence, if $0 \leq \omega_1$, then $x_0 < -\beta$ and by continuity $x(\omega_1, \epsilon) \leq -\beta/2$ for ϵ small. For future reference we compute

$$y_0 \equiv \lim_{\epsilon \to 0} y(\omega_1, \epsilon) = \omega_1[1 - \alpha_0 \cos \omega_0 h - \alpha_0 \gamma h \cos \omega_0 h + \alpha_0 \omega_0 h \sin \omega_0 h]$$

$$- \alpha_0 \gamma \sin \omega_0 h - \alpha_0 \omega_0 \cos \omega_0 h$$

$$= \omega_1 \left[\frac{\gamma(\gamma + \beta) + h\beta(\gamma^2 + \omega_0^2)}{\omega_0^2 + \gamma^2} \right] - \omega_0 \tag{13}$$

which, we note, can be zero for $\omega_1 > 0$.

In addition, we note that $|q_\epsilon(ik)| \geq \eta > 0$ for $k \neq \pm 1$ and ϵ small. For if $\lim_{n \to \infty} |q_\epsilon(ik_n)| = 0$, $k_n \neq \pm 1$, then for $n \geq N(\epsilon)$

$$k_n^2 \omega^2 = \frac{\alpha^2 \gamma^2 - \beta^2}{1 - \alpha^2} + O(\epsilon).$$

But $(\alpha^2\gamma^2 - \beta^2)/(1 - \alpha^2) = \omega_0^2 + O(\epsilon)$ and, therefore,

$$k_n^2(\omega_0 + \epsilon\omega_1)^2 = \omega_0^2 + O(\epsilon), \qquad k_n \neq \pm 1$$

which is impossible if ϵ is small.

Now we want to show that if $v(t)$ is a periodic solution of (4), then it is necessary and sufficient that $v(t)$ is an H_1^* solution of the following nonlinear integral equation

$$v(t) = \frac{1}{2\pi} \int_0^{2\pi} K^*(t - s) f(v(s))\, ds \tag{14}$$

where

$$K^*(t - s) = \sum_k \frac{\epsilon \exp(ik(t - s))}{q_\epsilon(ik)}. \tag{15}$$

We note that Eq. (14) has a free parameter ω_1 occurring in $q_\epsilon(ik)$.

To show necessity, let $v(t)$ be a real periodic solution of (4). Then v and $f(v)$ (since f is differentiable) have Fourier expansions

$$v = \sum_k v_k e^{ikt}$$

$$f(v) = \sum_k f_k e^{ikt},$$

where $v_k = \bar{v}_k$, $f_k = \bar{f}_k$. Multiplying (4) by $e^{-ikt}/2\pi$ and integrating, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} L_\epsilon[v]\, dt = \frac{\epsilon}{2\pi} \int_0^{2\pi} f(v(t)) e^{-ikt}\, dt$$

or

$$v_k q_\epsilon(ik) = \epsilon f_k .$$

Since $|q_\epsilon(ik)/\epsilon|$ is bounded away from zero for small ϵ, then we can solve for v_k;

$$v_k = \frac{\epsilon f_k}{q_\epsilon(ik)},$$

*The spaces H_0 and H_1 denote Hilbert spaces with metrics $||w||_0^2 = (1/2\pi) \int_0^{2\pi} w^2\, dt$ and $||w||_{0+1}^2 = ||w||_0^2 + ||w||_1^2$, respectively, where the notation $||w||_1^2$ will be used for the pseudo-metric $(1/2\pi) \int_0^{2\pi} (dw/dt)^2\, dt$.

and, therefore, v is the convolution of f and K^* given by (15); i.e.,

$$v(t) = \frac{1}{2\pi} \int_0^{2\pi} K^*(t - s) f(v(s)) \, ds. \tag{16}$$

To prove sufficiency, let v be an H_1 solution of (16). Then, v is continuously differentiable for the following reasons. First, $v \in H_1 \Rightarrow f(v) \in H_1$ since f is continuously differentiable. Hence, $\sum_k |ikf_k|^2 < \infty$. Differentiating (16) twice, we have

$$v''(t) = \frac{1}{2\pi} \int_0^{2\pi} \sum_k \frac{-\epsilon k^2 \exp(ik(t - s))}{q_\epsilon(ik)} f(v(s)) \, ds$$

which is in H_0 since for large k, $|1/q_\epsilon(ik)| = O(1/|k|)$, and hence,

$$||v''||_0^2 = \sum_k \left| \frac{\epsilon k^2 f_k}{q_\epsilon(ik)} \right|^2 < c_1 \sum |kf_k|^2 < \infty$$

provided ϵ is small. Thus, $v(t) \in H_2$, which implies that $v(t)$ is continuously differentiable. Note that by continuing we could get that $v \in H_{k+1}$ if $f \in C_k$.

Hence, we may operate with L_ϵ on v which yields

$$\begin{aligned} L_\epsilon[v] &= \frac{1}{2\pi} \int_0^{2\pi} \sum_k \frac{\epsilon}{q_\epsilon(ik)} L_\epsilon[\exp(ik(t - s))] f(v(s)) \, ds \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} \sum_k \exp(ik(t - s)) f(v(s)) \, ds \\ &= \epsilon f(v(t)); \end{aligned}$$

i.e., v is a solution of (4).

We now rewrite Eq. (16) as three integral equations. For any real periodic function $v(t)$ of periodic 2π, let

$$v = ae^{it} + \bar{a}e^{-it} + u(t), \tag{17}$$

where a is some complex number and u is orthogonal to $e^{\pm it}$, i.e., $(1/2\pi) \int_0^{2\pi} u(t) e^{\pm it} \, dt = 0$. We may assume that a is real since $v(t)$ is a solution if and only if $v(t + \phi)$ is also a solution. Thus, equation (16) can be decomposed into the following simultaneous integral equations,

$$ax = \frac{1}{2\pi} \int_0^{2\pi} f(v(t)) \cos t \, dt, \tag{18}$$

$$ay = -\frac{1}{2\pi} \int_0^{2\pi} f(v(t)) \sin t \, dt, \tag{19}$$

$$u(t) = \frac{\epsilon}{2\pi} \int_0^{2\pi} K(t - s) f(v(s)) \, ds, \tag{20}$$

where

$$K(t - s) = \sum_{k \neq \pm 1} \frac{\exp(ik(t - s))}{q_\epsilon(ik)}, \tag{21}$$

and v is given by (17) where a is real. Here, $x + iy \equiv q_\epsilon(i)/\epsilon$, as before. The problem is to prove for small ϵ the existence of a solution $(a, u(t), \omega)$. This will be done in two steps. First, we obtain an H_1 solution, $u(t, a, \omega_1)$, of Eq. (20) where $\omega = \omega_0 + \epsilon\omega_1$, a and ω_1 are considered free parameters, $\omega_1 \geq 0$, and $|a - a^*| < \frac{1}{2}|a^*|$. Then, we use the implicit function theorem to prove the existence of a, ω_1 such that Eqs. (18) and (19) are also satisfied.

The existence $u(t\ a, \omega_1) \in H_1$ of (20) will be shown using the following iteration

$$u_0(t, a, \omega_1) = 0,$$

$$u_{n+1} = \frac{\epsilon}{2\pi}\int_0^{2\pi} K(t-s)f(2a\cos s + u_n(s))\,ds, \qquad n = 0, 1, \cdots. \tag{22}$$

The proof is by induction with the following induction hypotheses for $n = 1, 2, 3, \cdots$,

$$(a) \qquad \max_{0\leq t\leq 2\pi} |u_n(t)| < O(\epsilon) \text{ (independent of } n),$$

$$\text{(b)*} \quad ||u_{n+1} - u_n||_{0,1}^2 < \tfrac{1}{4}\,||u_n - u_{n-1}||_0^2\,.$$

The first hypothesis guarantees that $v_n(t) = 2a\cos t + u_n(t)$ is bounded in the maximum norm; i.e., $\max_{0\leq t\leq 2\pi}|v_n(t)| < 2|a| + O(\epsilon) < 4|a^*|$; and hence, by assumption, $|f'(v_n)| < \nu$. The second hypothesis implies the convergence of the sequence u_n in H_1 since

$$||u_{n+1} - u_n||_{0,1} \leq (\tfrac{1}{2})^{n-1}\,||u_2 - u_1||_0$$

or

$$||u_n - u_m||_{0,1} < 2\,||u_{m+1} - u_m||_0 \leq (\tfrac{1}{2})^{m-2}\,||u_2 - u_1||_0$$

which approaches zero as $m, n \to \infty$.

We use the following

Lemma 2. Let $\mathcal{K}\phi = (1/2\pi)\int_0^{2\pi} K(t-s)\phi(s)\,ds$ where

$$K(t-s) = \sum_{k\neq\pm1} \frac{e^{ik(t-s)}}{q_\epsilon(ik)}$$

and ϕ is 2π-periodic. Then

$$||\mathcal{K}\phi||_{0,1}^2 \leq c^2\,||\phi||_0^2\,,$$

where

$$c = \max_{k\neq\pm1}\left|\frac{ik}{q_\epsilon(ik)}\right| < \infty.$$

Proof: If ϕ_k denote the Fourier coefficients of ϕ, then by Parseval's equality,

$$||\mathcal{K}\phi||_0^2 = \sum_{k\neq\pm1}\left|\frac{\phi_k}{q_\epsilon(ik)}\right|^2 \leq c_0^2\,||\phi||_0^2\,,$$

where $c_0 = \max_{k\neq\pm1}|1/q_\epsilon(ik)|$. Similarly,

$$||\mathcal{K}\phi||_1^2 = \sum_{k\neq\pm1}\left|\frac{ik\phi_k}{q_\epsilon(ik)}\right|^2 \leq c^2\,||\phi||_0^2\,,$$

where $c = \max_{k\neq\pm1}|ik/q_\epsilon(ik)| \geq c_0$. It remains to be proved that c is finite, but this follows since $|q_\epsilon(ik)| \geq \eta > 0$ for ϵ small, $k \neq \pm1$, and $1/|q_\epsilon(ik)| = O(|k|^{-1})$, $|k| \to \infty$. This completes the proof of Lemma 2.

*The notation $||w||_{0,1}^2 < g$ denotes $||w||_0^2 < g$ and $||w||_1^2 < g$.

To prove (a) for $n = 1$, we use the fact that

$$\max_{0 \le t \le 2\pi} |w(t)| \le (2\pi)^{3/2}(||w||_0 + ||w||_1). \tag{23}$$

From Lemma 2 and (22) we have

$$||u_1||_{0,1} \le \epsilon c \, ||f(2a \cos t)||_0 ,$$

and since we assume $|a - a^*| < \frac{1}{2} |a^*|$, then $|2a \cos t| < 3 \, |a^*|$, $|f'| < \nu$, and hence (since $f(0) = 0$),

$$||u_1||_{0,1} \le 3\epsilon c\nu \, |a^*|.$$

Thus, by (23),

$$\max_{0 \le t \le 2\pi} |u_1(t)| \le O(\epsilon),$$

proving (a) for $n = 1$.

To prove (b) for $n = 1$, we compute

$$u_2 - u_1 = \frac{\epsilon}{2\pi} \int_0^{2\pi} K(t - s)[f(v_1(s) - f(v_0(s))] \, ds,$$

where $v_n(s) = 2a \cos s + u_n(s)$. By Lemma 2 and the fact that $|v_1| < 4 \, |a^*|$, $|v_0| < 4 \, |a^*|$, we have

$$||u_2 - u_1||_{0,1}^2 \le (\epsilon c\nu)^2 \, ||u_1 - u_0||_0^2 < \tfrac{1}{4} \, ||u_1 - u_0||_0^2$$

for $\epsilon < \frac{1}{2} c\nu$ proving (b) for $n = 1$.

Now assume (a) and (b) hold for $n \le N - 1$. Then, by (*b*)

$$||u_n - u_{n-1}||_{0,1}^2 \le \tfrac{1}{4} \, ||u_{n-1} - u_{n-2}||_0^2 , \qquad n = 2, 3, \cdots, N$$

and by (23)

$$\begin{aligned}
\max_{0 \le t \le 2\pi} |u_N(t)| &\le (2\pi)^{3/2}(||u_N||_0 + ||u_N||_1) \\
&\le (2\pi)^{3/2} \sum_{n=1}^{N} (||u_n - u_{n-1}||_0 + ||u_n - u_{n-1}||_1) \\
&\le (2\pi)^{3/2}\left[2 \sum_{n=2}^{N} (1/2)^{n-1} \, ||u_1||_0 + ||u_1||_0 + ||u_1||_1\right] \\
&\le 4(2\pi)^{3/2} 3\epsilon\nu \, |a^*| \\
&\le O(\epsilon).
\end{aligned}$$

Thus, (a) holds for $n = N$.

Using (a) we have $|v_N(t)| \le 4 \mid a^*|$ which gives us that $|f'(v_n)| < \nu$. Then by Lemma 2

$$\begin{aligned}
||u_{N+1} - u_N||_{0,1}^2 &\le (\epsilon c)^2 \, ||f(v_N) - f(v_{N-1})||_0^2 \\
&\le (\epsilon c\nu)^2 \, ||u_N - u_{N-1}||_0^2 \\
&\le \tfrac{1}{4} \, ||u_N - u_{N-1}||_0^2
\end{aligned}$$

proving (b) for $n = N$. By induction (a) and (b) hold for all n proving the convergence of u_n in H_1 .

The last part of the proof consists of proving that for ϵ small there exist $a(\epsilon)$, $\omega_1(\epsilon)$ such that Eqs. (18) and (19) hold. The proof uses the implicit function theorem.

Define

$$F(a, \omega_1, \epsilon) \equiv ax(\omega_1, \epsilon) - \frac{1}{2\pi}\int_0^{2\pi} f(2a \cos t + u^*(t, a, \omega_1, \epsilon)) \cos t \, dt,$$

and

$$G(a, \omega_1, \epsilon) \equiv ay(\omega_1, \epsilon) + \frac{1}{2\pi}\int_0^{2\pi} f(2a \cos t + u^*(t, a, \omega_1, \epsilon)) \sin t \, dt,$$

where $u^*(t, a, \omega_1, \epsilon)$ is the solution of Eq. (16). Equations (18) and (19) are simply

$$F(a, \omega_1, \epsilon) = 0,$$
$$G(a, \omega_1, \epsilon) = 0.$$

For $\epsilon = 0$, we have the solution

$$a(0) = a^*,$$
$$\omega_1(0) = \frac{\omega_0(\omega_0^2 + \gamma^2)}{\gamma(\gamma + \beta) + h\beta(\gamma^2 + \omega_0^2)}. \tag{24}$$

This follows since $u^*(t, a, \omega_1, 0) = 0$, so that

$$F(a^*, \omega_1, 0) = a^*x_0 - \frac{1}{2\pi}\int_0^{2\pi} f(2a^* \cos t) \cos t \, dt = 0$$

by (10). Similarly,

$$G(a^*, \omega_1, 0) = a^*y_0 + \frac{1}{2\pi}\int_0^{2\pi} f(2a^* \cos t) \sin t \, dt = a^*y_0$$

where we have used the fact that $f(2a^* \cos t)$ is even function of t. With the choice of ω_1 given by (24) and using (13) we see that $y_0 = 0$.

It remains to be shown that

$$\left.\begin{vmatrix} \dfrac{\partial F}{\partial a} & \dfrac{\partial F}{\partial \omega_1} \\[2ex] \dfrac{\partial G}{\partial a} & \dfrac{\partial G}{\partial \omega_1} \end{vmatrix}\right|_{a=a^*;\ \omega_1=\omega_1(0);\ \epsilon=0} \neq 0. \tag{25}$$

However, there still exists the question whether these derivatives exist. The argument for this will be omitted since it provides nothing interesting. The basic argument consists of differentiating Eq. (20) with respect to a and ω_1 and showing by the contracting map theorem that there exist unique solutions of the resulting equations. Then one can show that the difference quotients

$$\frac{u^*(t, a + \Delta a, \omega_1, \epsilon) - u^*(t, a, \omega_1, \epsilon)}{\Delta a}$$

and

$$\frac{u^*(t, a, \omega_1 + \Delta\omega_1, \epsilon) - u^*(t, a, \omega_1, \epsilon)}{\Delta\omega_1}$$

are uniformly bounded in H_1 and equicontinuous, and, therefore, have a convergent subsequence. However, the limit must satisfy the differentiated equation and hence is unique.

To prove (25), we note that

$$\left.\frac{\partial u^*}{\partial a}\right|_{\epsilon=0} = \left.\frac{\partial u^*}{\partial \omega_1}\right|_{\epsilon=0} = 0$$

since

$$u^*(t) = \frac{\epsilon}{2\pi}\int_0^{2\pi} K(t-s)f(2a\cos s + u^*(s))\,ds$$

and

$$\left.\frac{\partial K}{\partial \omega_1}\right|_{\epsilon=0} = 0.$$

We compute

$$\left.\frac{\partial F}{\partial a}\right|_{a=a^*;\,\omega_1=\omega_1(0);\,\epsilon=0} = x_0 - \frac{1}{2\pi}\int_0^{2\pi} f'(2a^*\cos t)2\cos^2 t\,dt, \tag{26}$$

$$\left.\frac{\partial G}{\partial \omega_1}\right|_{a=a^*;\,\omega_1=\omega_1(0);\,\epsilon=0} = a^*\frac{\omega_0}{\omega_1(0)},$$

$$\left.\frac{\partial G}{\partial a}\right|_{a=a^*;\,\omega_1=\omega_1(0);\,\epsilon=0} = y_0 + \frac{1}{2\pi}\int_0^{2\pi} f'(2a^*\cos t)2\cos t\sin t\,dt = 0.$$

Since (26) is not zero by assumption and $a^*\omega_0/\omega_1(0) \neq 0$, we have that (25) is also not zero. By the implicit function theorem for small ϵ there exist unique functions $\omega_1 = \omega_1(\epsilon)$ and $a = a(\epsilon)$ such that

$$F(a(\epsilon), \omega_1(\epsilon), \epsilon) \equiv 0,$$

$$G(a(\epsilon), \omega_1(\epsilon), \epsilon) \equiv 0,$$

and $a(0) = a^*$. This completes the proof of the theorem.

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References

1. R. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.
2. N. Krasovskii, *On periodic solutions of differential equations involving a time lag*, Dokl. Acad. Nauk (N. S.), **114** 252-255 (1957).
3. N. Shimanov, *Almost periodic solutions in nonlinear systems with retardation*, Dokl. Acad. Nauk, S. S. S. R., **125** 1203-1206 (1959).
4. N. Shimanov, *On the vibration theory of quasilinear systems with lags*, Prikl. Mat. Meh. **23** 836-844 (1959); TPMM, 1198-1208.
5. J. K. Hale, *Linear functional-differential equations with constant coefficients*, Contributions to Differential Equations (2), 291-317 (1963).
6. R. K. Brayton, *Nonlinear oscillations in a distributed network*. To appear in a forthcoming issue of Quart. of Applied Math.