SOLUTION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS*

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1. Introduction. Consider the two integral equations

$$\int_0^1 F(y)\,|x-y|^{-\alpha}\,dy = f(x), \qquad 0 < \alpha < 1 \tag{1}$$

and

$$\int_0^1 F(y)\,|x-y|^{-\alpha}\,\mathrm{sgn}\,(x-y)\,dy = f(x), \qquad 0 < \alpha < 1 \tag{2}$$

differing from one another only in the occurrence of the algebraic sign of $(x - y)$ in the kernel of Eq. (2). Equation (1) has been solved by Carleman [1]. The solution of Eq. (2), which has not been given in the literature, will be obtained here by a double use of the Wiener-Hopf technique ([2] is an easy reference). It will be shown that the formal limit as $\alpha \to 1$ in the solution of Eq. (2) is the same as the solution of

$$\text{P.V.}\int_0^1 F(y)(x-y)^{-1}\,dy = f(x) \tag{3}$$

which has been treated by Muskhelishvili [3] and others.

In Section 4 the solution of Carleman's equation will be obtained.

Both of these integral equations occur (with $\alpha = \frac{1}{2}$) in the physical literature [4] in connection with Magnetohydrodynamic flow in a rectangular duct when the walls parallel to the applied magnetic field are perfectly conducting. This is discussed in Section 5.

In Section 6 it will be shown how the solution of the more general equation,

$$\int_0^1 F(y)\,|x-y|^{m-\alpha}\,[\mathrm{sgn}\,(x-y)]^n\,dy = f(x)$$

may be obtained from Eqs. (1) and (2). Here m is a positive integer and n is zero or one.

2. The solution of $\int_0^\infty F(y)\,|x-y|^{-\alpha}\,\mathrm{sgn}\,(x-y)\,dy = f_1(x)$. We will proceed in two steps. We will first solve

$$\int_0^\infty F(y)\,|x-y|^{-\alpha}\,\mathrm{sgn}\,(x-y)\,dy = f_1(x) \tag{4}$$

with $f_1(x)$ given on the interval from zero to infinity, $f_1(x) = f(x)$ on [0, 1]. Then the condition $F(y) = 0$ when $y > 1$ will give an integral equation for $f_1(x)$ when $x > 1$ and will lead to the solution of Eq. (2).

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We will modify Eq. (4) slightly by introducing a convergence factor which will separate branch cuts in Laplace transform space. The modified equation is

$$\int_0^\infty F(y)\,|x-y|^{-\alpha}\,\mathrm{sgn}\,(x-y)\exp(-\epsilon\,|x-y|)\,dy = f_1(x)H(x) + f_2(x)H(-x) \tag{5}$$

where $H(x)$ is Heaviside's step function defined by

$$H(x) = 1, \qquad x > 0,$$
$$= 0, \qquad x < 0,$$

the function $f_2(x)$ is defined by the integral term when x is negative. Now let us take the two-sided Laplace transform [5] of Eq. (5) with

$$y^*(s) = \int_{-\infty}^{\infty} e^{-sx} y(x)\,dx.$$

The result of this operation is

$$F^*(s)_+\Gamma(1-\alpha)[(\epsilon+s)^{\alpha-1} - (\epsilon-s)^{\alpha-1}] = f_1^*(s)_+ + f_2^*(s)_- \tag{6}$$

where

$$F^*(s)_+ = \int_0^\infty e^{-sx}F(x)\,dx,$$

$$f_1^*(s)_+ = \int_0^\infty e^{-sx}f_1(x)\,dx,$$

$$f_2^*(s)_- = \int_{-\infty}^0 e^{-sx}f_2(x)\,dx.$$

The complex functions $(\epsilon + s)^{\alpha-1}$ and $(\epsilon - s)^{\alpha-1}$ are principal branches, defined to be real when their arguments are real and positive, with branch cuts along the lines in s space where their arguments are real and negative. This leaves a strip $-\epsilon < \mathrm{Re}\, s < \epsilon$ where the bracketed term is analytic.

We will place the following restrictions on the functions $F(x)$, $f_1(x)$, $f_2(x)$.

(i) $F(x) = Oe^{-\epsilon x}$ as $x \to \infty$, and $F(x) = Ox^{-p}$, $p < 1$ as $x \to 0$. The latter condition is required in order that the integral in the integral equation be convergent. With these conditions, we conclude that $F^*(s)_+$ is analytic for $\mathrm{Re}\, s > -\epsilon$ and

$$|F^*(s)_+| = O\,|s|^{p-1} \quad \text{as} \quad |s| \to \infty.$$

(ii) Similarly we assume $f_1(x) = Oe^{-\epsilon x}$ as $x \to \infty$ and $f_1(x) = O1$ as $x \to 0$ so that $f_1^*(s)_+$ will be analytic for $\mathrm{Re}\, s > -\epsilon$ and $|f_1^*(s)_+| = O\,|s|^{-1}$ as $|s| \to \infty$.

(iii) Also we assume $f_2(x) = Oe^{\epsilon x}$ as $x \to -\infty$ and $f_2(x) = O(-x)^{-q}$, $q < 1$ as $x \to 0$. Then $f_2^*(s)_-$ will be analytic for $\mathrm{Re}\, s < \epsilon$ and $|f_2^*(s)_-| = O\,|s|^{-1+q}$ as $|s| \to \infty$. Based on these assumptions both sides of Eq. (6) are analytic in the strip $-\epsilon < \mathrm{Re}\, s < \epsilon$.

In order to use the Weiner-Hopf technique, we want to rearrange Eq. (6) so that the right and left sides of the equation are analytic in overlapping half planes. To this end we use Cauchy's Integral formula to decompose the factor $(\epsilon + s)^{\alpha-1} - (\epsilon - s)^{\alpha-1}$ as follows. The function

$$g(s) = [(\epsilon+s)^{\alpha-1} - (\epsilon-s)^{\alpha-1}](\epsilon+s)^{1-\alpha/2}(\epsilon-s)^{-\alpha/2}/2\sin(1-\alpha)\pi/2$$

is analytic and has no zeros in the strip $-\epsilon < \text{Re } s < 0$, further $g(s) \to 1$ as $s \to \infty$, in this strip. (One has to be careful of the branches to show this last property.) Therefore $\ln g(s)$ is analytic in this strip and tends to zero at infinity. By Cauchy's formula we can write

$$\ln g(s) = \frac{1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \ln g(w) \frac{dw}{w - s} - \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \ln g(w) \frac{dw}{w - s}$$

$$= \tau(s)_- - \tau(s)_+$$

with $-\epsilon < \gamma_1 < \gamma_2 < 0$. We see that τ_- is analytic for $\text{Re } s < \gamma_2$, τ_+ is analytic for $\text{Re } s > \gamma_1$. Also it can be shown that τ_+ and τ_- both tend to zero as $\epsilon \to 0$ and as $|s| \to \infty$. Then $g(s) = \exp(\tau_-) \exp(-\tau_+)$, and

$$(\epsilon + s)^{\alpha - 1} - (\epsilon - s)^{\alpha - 1}$$
$$= 2 \sin (1 - \alpha)\pi/2[(\epsilon + s)^{-1+\alpha/2} \exp(-\tau_+)][(\epsilon - s)^{\alpha/2} \exp(\tau_-)]. \tag{7}$$

So we have decomposed this difference into the product of functions analytic in overlapping left and right half planes.

Substituting Eq. (7) into Eq. (6) gives

$$h(\alpha)F^*(s)_+(\epsilon + s)^{-1+\alpha/2} \exp(-\tau_+) = (f_1^*(s)_+ + f_2^*(s)_-)(\epsilon - s)^{-\alpha/2} \exp(-\tau_-), \tag{8}$$

where $h(\alpha) = 2 \sin (1 - \alpha)\pi/2\Gamma(1 - \alpha)$. The left hand side is analytic for $\text{Re } s > \gamma_1$, and the second term on the right is analytic for $\text{Re } s < \gamma_2$. The first term on the right, $Q(s) = f_1^*(s)_+(\epsilon - s)^{-\alpha/2} \exp(-\tau_-)$, is analytic in the strip $\gamma_1 < \text{Re } s < \gamma_2$. Using Cauchy's formula again, we can write this as

$$Q(s) = \frac{1}{2\pi i} \int_{\gamma_2' - i\infty}^{\gamma_2' + i\infty} Q(w) \frac{dw}{w - s} - \frac{1}{2\pi i} \int_{\gamma_1' - i\infty}^{\gamma_1' + i\infty} Q(w) \frac{dw}{w - s}$$

$$= Q(s)_- - Q(s)_+$$

where $-\epsilon < \gamma_1 < \gamma_1' < \gamma_2' < \gamma_2 < 0$. $Q(s)_-$ is analytic when $\text{Re } s < \gamma_2'$ and $Q(s)_+$ is analytic when $\text{Re } s > \gamma_1'$. Substituting this into Eq. (8) yields

$$h(\alpha)F^*(s)_+(\epsilon + s)^{-1+\alpha/2} \exp(-\tau_+) + Q(s)_+$$
$$= Q(s)_- + f_2^*(s)_-(\epsilon - s)^{-\alpha/2} \exp(-\tau_-) = P(s). \tag{9}$$

Since the left and right sides are analytic in the overlapping half planes $\text{Re } s > \gamma_1'$ and $\text{Re } s < \gamma_2'$ respectively, they define an entire function $P(s)$ analytic in the whole s plane. By the estimates made in (i), (ii), and (iii) we find

$$|F^*(s)_+(\epsilon + s)^{-1+\alpha/2} \exp(-\tau_+)| = O\, |s|^{p-2+\alpha/2}, \qquad p < 1$$

as $|s| \to \infty$ with $\text{Re } s > \gamma_1'$, and

$$|f_2^*(s)_-(\epsilon - s)^{-\alpha/2} \exp(-\tau_-)| = O\, |s|^{q-1-\alpha/2}$$

as $|s| \to \infty$ with $\text{Re } s < \gamma_2'$. Both of these functions tend to zero at infinity. It can also be shown that Q_+ and Q_- tend to zero at infinity. Therefore by Liouville's Theorem, the entire function $P(s)$ must be zero. Consequently Eq. (9) shows that

$$h(\alpha)F^*(s)_+ = -Q(s)_+(\epsilon + s)^{1-\alpha/2} \exp(\tau_+).$$

Now with Re $s > 0$ let us take the limit as $\epsilon \to 0$. We get

$$h(\alpha)F^*(s)_+ = -s^{1-\alpha/2}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f_1^*(w)_+(-w)^{-\alpha/2}\frac{dw}{w-s}, \tag{10}$$

where the integration is taken along the imaginary axis. Taking the inverse transform of Eq. (10) (App. 1) yields

$$h(\alpha)F(x) = \frac{1}{\Gamma^2(\alpha/2)}\frac{d}{dx}\int_0^x (x-\beta)^{-1+\alpha/2}\,d\beta\int_\beta^\infty f_1(\gamma)(\gamma-\beta)^{-1+\alpha/2}\,d\gamma. \tag{11}$$

This is the solution of Eq. (4).

3. The solution of Eq. (2). In Eq. (2), $f(x)$ is only specified for x between zero and one, while $F(x)$ is desired on the same interval. Therefore let us try to select $f_1(x)$ for $x > 1$ in Eq. (11) in such a way that $F(x)$ will be zero when $x > 1$. Then $F(x)$, will be the solution of Eq. (2). In Eq. (11) let

$$\begin{aligned} F(x) &= F(x)H(1-x), \\ f_1(x) &= f(x)H(1-x) + f_3(x)H(x-1). \end{aligned} \tag{12}$$

Then Eq. (11) is an integral equation for the unknown functions $F(x)$ and $f_3(x)$.

To solve this integral equation, the same procedure is followed as in Section 2, using the Mellin transform ([6, Vol. 1, p. 307]) in place of the two-sided Laplace transform. Operating on both sides of Eq. (11) with the Mellin operator, defined by $M_s(y(x)) = \int_0^\infty x^{s-1}y(x)\,dx$, yields the relation ([6, Vol. 2, p. 183])

$$h(\alpha)M_s(F(x)) = -\frac{(s-1)\Gamma(2-\alpha/2-s)\Gamma(s-1+\alpha/2)}{\Gamma(2-s)\Gamma(s-1+\alpha)}M_s(x^{\alpha-1}f_1(x)). \tag{13}$$

It is the existence of this relation which allows us to use function theoretic methods. Nothing so simple results when the Kernel in Eq. (2) is not a power of $|x-y|$. The same operation on Eqs. (12) gives

$$M_s(F(x)) = F^{**}(s)_+ = \int_0^1 F(x)x^{s-1}\,dx,$$

$$\begin{aligned} M_s(x^{\alpha-1}f_1(x)) &= M_s(x^{\alpha-1}f(x)H(1-x)) + M_s(x^{\alpha-1}f_3(x)H(x-1)) \\ &= T(s)_+ + T(s)_- . \end{aligned}$$

Substituting these into Eq. (13) gives, after some rearrangements,

$$h(\alpha)F^{**}(s)_+\frac{\Gamma(s-1+\alpha)}{\Gamma(s-1+\alpha/2)} = -\frac{(s-1)\Gamma(2-\alpha/2-s)}{\Gamma(2-s)}(T(s)_+ + T(s)_-). \tag{14}$$

As in Section 2, we place the following restrictions on $F(x)$, $f(x)$ and $f_3(x)$.

(i) We assume $F(x) = Ox^{-p}$, $p < 1$ as $x \to 0$ and $F(x) = O(1-x)^{-t}$, $t < 1$ as $x \to 1$. Then $F^{**}(s)_+$ will be analytic for Re $s > p$. The asymptotic behavior of $F^{**}(s)_+$ can be seen by substituting $x = e^{-z}$ in the defining integral, reducing it to a Laplace integral. It is easily seen then that $|F^{**}(s)_+| = O\,|s|^{-1+t}$ as $|s| \to \infty$.

(ii) We assume $f(x) = O1$ as $x \to 0$ and as $x \to 1$. Then $T(s)_+$ will be analytic for Re $s > 1-\alpha$ and $|T(s)_+| = O\,|s|^{-1}$ as $|s| \to \infty$.

(iii) We assume $f_3(x) = Ox^{-\alpha}$, as $x \to \infty$ and $f_3(x) = O(x - 1)^{-u}$, $u < 1$ as $x \to 1$. Therefore $T(s)_-$ is analytic for Re $s < 1$ and $|T(s)_-| = O\,|s|^{-1+u}$ as $|s| \to \infty$.

(iv) The behavior of the ratios of Gamma functions may be found by substituting $x = e^{-\sigma}$ into the Beta function representation,

$$\frac{\Gamma(s)}{\Gamma(s+\nu)} = \frac{1}{\Gamma(\nu)} \int_0^1 x^{s-1}(1-x)^{\nu-1}\,dx.$$

From this it is easily seen that $\Gamma(s - 1 + \alpha)/\Gamma(s - 1 + \alpha/2)$ is analytic for Re $s > 1 - \alpha$ and is of order $|s|^{\alpha/2}$ as $|s| \to \infty$. While $\Gamma(2 - \alpha/2 - s)/\Gamma(2 - s)$ is analytic for Re $s < 2 - \alpha/2$ and is of order $|s|^{-\alpha/2}$ as $|s| \to \infty$.

A little reflection on the regions of analyticity of the various functions involved shows that the lefthand side of Eq. (14) is analytic for Re $s > 1 - \alpha$ or Re $s > p$ whichever is larger. Since $\alpha > 0$ and $p < 1$ there is some number $\nu < 1$ such that $\nu > \max(1 - \alpha, p)$. Then the lefthand side is analytic for Re $s > \nu$, $\nu < 1$. The second term on the right of Eq. (14) is analytic for Re $s < 1$. The first term on the right,

$$R(s) = -(s-1)T(s)_+\Gamma(2 - \alpha/2 - s)/\Gamma(2 - s),$$

is analytic for $1 - \alpha <$ Re $s < 2 - \alpha/2$. Using Cauchy's Formula in this strip we can write

$$\begin{aligned} R(s) &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} R(w)\,\frac{dw}{w-s} - \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} R(w)\,\frac{dw}{w-s} \\ &= R(s)_- - R(s)_+\,. \end{aligned} \tag{15}$$

$R(s)_-$ is analytic for Re $s < 1$ and $R(s)_+$ is analytic for Re $s > \nu$, $\nu < 1$. Both functions tend to zero as $|s| \to \infty$. Substituting Eq. (15) into Eq. (14) gives

$$\begin{aligned} &h(\alpha)F^{**}(s)_+\Gamma(s - 1 + \alpha)/\Gamma(s - 1 + \alpha/2) + R(s)_+ \\ &\qquad = R(s)_- - (s-1)T(s)_-\Gamma(2 - \alpha/2 - s)/\Gamma(2 - s) = E(s). \end{aligned} \tag{16}$$

Since the left hand side is analytic for Re $s > \nu$ and the righthand side is analytic in the overlapping half plane Re $s < 1$ the two sides of Eq. (16) define the entire function $E(s)$. By the estimates made above, the left hand side is $O\,|s|^{-1+t+\alpha/2}$, $t < 1$ as $|s| \to \infty$. The right hand side is $O\,|s|^{u-\alpha/2}$, $u < 1$ as $|s| \to \infty$. We cannot assert, without restricting t and u that $E(s) \to 0$ at infinity. However, dE/ds does tend to zero as $|s| \to \infty$. Therefore the entire function $E(s)$ is at most a constant, C.

From Eq. (16) we have the result

$$h(\alpha)F^{**}(s)_+ = (C - R(s)_+)\Gamma(s - 1 + \alpha/2)/\Gamma(s - 1 + \alpha). \tag{17}$$

Before taking the inverse transform of Eq. (17), let us check the assumptions made in (i) of this section. We see from Eq. (17) that $F^{**}(s)_+$ is analytic for Re $s > \max(\nu, 1-\alpha/2)$, but ν is greater than $\max(p, 1 - \alpha)$ and the assumption of (i) was that $F^{**}(s)_+$ be analytic for Re $s > p$. Therefore $p = \max(p, 1 - \alpha, 1 - \alpha/2)$ which is satisfied by $p = 1 - \alpha/2$. So $p < 1$ as assumed. Also $|F^{**}(s)_+| = O\,|s|^{-\alpha/2}$, which is consistent with the other assumption of (i), namely $|F^{**}(s)_+| = O\,|s|^{-1+t}$, $t < 1$.

We can also check the less obvious assumptions made in (iii). From Eq. (16)

$$T(s)_- = \frac{\Gamma(2-s)(R(s)_- - C)}{(s-1)\Gamma(2 - \alpha/2 - s)}.$$

Because of the pole at $s = 1$, $T(s)_-$ is only analytic for Re $s < 1$ as assumed and we could not have stretched this to the right. Also $|T(s)_-| = O\,|s|^{-1+\alpha/2}$, which is consistent with the order estimate in (iii).

The inversion of Eq. (17) is similar to the Laplace inversion of Eq. (10) using the Mellin convolution ([6, Vol. 1, p. 308]) in place of the Laplace convolution. The result is

$$F(x) = c_1 x^{-1+\alpha/2}(1 - x)^{-1+\alpha/2} + \frac{1}{h(\alpha)\Gamma^2(\alpha/2)} \frac{d}{dx} x^{\alpha/2} \int_x^1 \xi^{-\alpha}(\xi - x)^{-1+\alpha/2}\, d\xi \int_0^\xi \beta^{\alpha/2}(\xi - \beta)^{-1+\alpha/2} f(\beta)\, d\beta, \tag{18}$$

which is the solution of Eq. (2). It can be shown by direct integration that the first term in Eq. (18) is the solution of the homogeneous equation

$$\int_0^1 F(y)\,|x - y|^{-\alpha} \operatorname{sgn}(x - y)\, dy = 0.$$

There is no requirement that $f(x)$ be orthogonal to the solution of the homogeneous equation. This can be easily checked by taking $f(x) = 1$ (which does not satisfy the orthogonality conditions) in Eq. (18). The result is

$$F(x) = \frac{\sin \alpha\pi/2}{\pi} x^{-1+\alpha/2}(1 - x)^{-1+\alpha/2}(c_2 - x),$$

which can be shown to satisfy Eq. (2). In this respect, Eq. (2) resembles a singular integral equation with Cauchy type kernel. In fact if we take the limit as $\alpha \to 1$ Eq. (18) tends to a sensible limit, namely

$$F(x) = c_1 x^{-1/2}(1 - x)^{-1/2} + \frac{1}{\pi^2} \frac{d}{dx} x^{1/2} \int_x^1 \xi^{-1}(\xi - x)^{-1/2}\, d\xi \int_0^\xi \beta^{1/2}(\xi - \beta)^{-1/2} f(\beta)\, d\beta. \tag{19}$$

This is not the solution of Eq. (2) with $\alpha = 1$,

$$\int_0^1 F(y)(x - y)^{-1}\, dy = f(x), \tag{20}$$

since the kernel of this equation is not integrable. We will show that Eq. (19) is the solution of Eq. (20) if the integral in Eq. (20) is interpreted as a principal value.

After changing the order of integration and carrying out the integration on ξ, the second term on the right of Eq. (19) can be written

$$-\frac{1}{\pi^2} \frac{d}{dx} \int_0^1 f(\beta) \ln \left\{ x\beta \left(\left(\frac{1 - x}{x} \right)^{1/2} - \left(\frac{1 - \beta}{\beta} \right)^{1/2} \right)^2 \Big/ |\beta - x| \right\} d\beta.$$

The derivative can be taken inside the integral provided the result is interpreted as a principal value and $df/d\beta$ is bounded. This gives then

$$F(x) = c_1 x^{-1/2}(1 - x)^{-1/2} + \frac{1}{\pi^2} x^{-1/2}(1 - x)^{-1/2}\ \text{P.V.} \int_0^1 \frac{f(\beta)\beta^{1/2}(1 - \beta)^{1/2}}{\beta - x}\, d\beta \tag{21}$$

which is the solution of

$$\text{P.V.} \int_0^1 F(y)(x - y)^{-1}\, dy = f(x)$$

as is shown in [7, p. 131]. We conclude that if the integral in Eq. (2) is interpreted as a principal value, Eq. (18) gives the solution for $0 < \alpha \leq 1$.

4. The solution of Carleman's equation, Eq. (1). Equation (1) can be solved by the same methods used in Sections 2 and 3. Since the procedure is almost the same, we will merely show the essential steps. We begin by solving

$$\int_0^\infty F(y)\,|x-y|^{-\alpha}\exp(-\epsilon\,|x-y|)\,dy = f_1(x). \tag{22}$$

Taking the two-sided Laplace transform of this, we get

$$F^*(s)_+\Gamma(1-\alpha)[(\epsilon+s)^{\alpha-1}+(\epsilon-s)^{\alpha-1}] = f_1^*(s)_+ + f_2^*(s)_- \tag{23}$$

where the definitions and estimates of the functions involved are the same as in Section 2. The only difference is the plus sign in the bracketed term. Because of this difference, this term will factor differently. We find

$$\begin{aligned}(\epsilon+s)^{\alpha-1}&+(\epsilon-s)^{\alpha-1}\\ &= 2\cos(\alpha-1)\pi/2[(\epsilon+s)^{\alpha/2-1/2}\exp(-\tau'_+)][(\epsilon-s)^{\alpha/2-1/2}\exp(\tau'_-)]\end{aligned} \tag{24}$$

where

$$\tau'(s)_+ = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\ln g'(w)\,\frac{dw}{w-s},$$

$$\tau'(s)_- = \frac{1}{2\pi i}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\ln g'(w)\,\frac{dw}{w-s}$$

with

$$g'(s) = [(\epsilon+s)^{\alpha-1}+(\epsilon-s)^{\alpha-1}](\epsilon+s)^{1/2-\alpha/2}(\epsilon-s)^{1/2-\alpha/2}/2\cos(\alpha-1)\pi/2$$

The functions τ'_+ and τ'_- are, as before, analytic in right and left half-planes and tend to zero as $\epsilon\to 0$. Proceeding as before, we split the function

$$Q'(s) = f_1^*(s)_+(\epsilon-s)^{1/2-\alpha/2}\exp(-\tau'_-)$$

into the difference of two analytic functions $Q'_- - Q'_+$. The result is

$$\begin{aligned}h'(\alpha)F^*(s)_+(\epsilon+s)^{\alpha/2-1/2}&\exp(-\tau'_+) + Q'(s)_+\\ &= Q'(s)_- + f_2^*(s)_-(\epsilon-s)^{1/2-\alpha/2}\exp(-\tau'_-) = 0\end{aligned} \tag{25}$$

where $h'(\alpha) = 2\cos(\alpha-1)\pi/2\,\Gamma(1-\alpha)$. We find as before that the entire function is zero. This yields, when $\epsilon\to 0$,

$$h'(\alpha)F^*(s)_+ = ss^{-1/2-\alpha/2}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} wf_1^*(w)_+(-w)^{-1/2-\alpha/2}\frac{dw}{w-s}. \tag{26}$$

The inverse transform of this is

$$h'(\alpha)F(x) = -\Gamma^{-2}(\alpha/2+1/2)\frac{d}{dx}\int_0^x(x-\beta)^{\alpha/2-1/2}\,d\beta\,\frac{d}{d\beta}\int_\beta^\infty(\gamma-\beta)^{\alpha/2-1/2}f_1(\gamma)\,d\gamma \tag{27}$$

which is the solution of

$$\int_0^\infty F(y)\,|x-y|^{-\alpha}\,dy = f_1(x). \tag{28}$$

This should be compared with Eq. (11).

Now if we set $F(x) = 0$ when $x > 1$ in Eq. (27) and take the Mellin Transform, we get an equation analogous to Eq. (16).

$$h'(\alpha)F^{**}(s)_+ \frac{\Gamma(s-1+\alpha)}{\Gamma(s-1+\alpha/2+1/2)} + R'(s)_+$$
$$= R'(s)_- - (s-1)\frac{\Gamma(2-s-\alpha/2-1/2)}{\Gamma(2-s)} T(s)_- = E'(s) \qquad (29)$$

where $T(s)_+$ and $T(s)_-$ are defined the same as in section 3 and

$$R'(s)_- - R'(s)_+ = R'(s) = -(s-1)\frac{\Gamma(2-s-\alpha/2-1/2)}{\Gamma(2-s)} T(s)_+$$

as in Eq. (15). The asymptotic behavior of the functions occurring in Eq. (29) is the same as in Eq. (16) except for the ratios of Gamma functions which have different arguments. There is just enough difference that the entire function $E'(s)$ must be zero. Therefore

$$h'(\alpha)F^{**}(s)_+ = -R'(s)_+ \frac{\Gamma(s-1+\alpha/2+1/2)}{\Gamma(s-1+\alpha)} \qquad (30)$$

The inversion of this gives

$$h'(\alpha)\Gamma^2(\alpha/2+1/2)F(x)$$
$$= -x^{\alpha/2-1/2}\frac{d}{dx}\int_x^1 \xi^{1-\alpha}(\xi-x)^{\alpha/2-1/2}\,d\xi\,\frac{d}{d\xi}\int_0^\xi \eta^{-1/2+\alpha/2}(\xi-\eta)^{-1/2+\alpha/2}f(\eta)\,d\eta \qquad (31)$$

which is the solution of Eq. (1),

$$\int_0^1 F(y)\,|x-y|^{-\alpha}\,dy = f(x).$$

The solution given above is not quite in the same form as that given by Carleman [1]. In the case $\alpha = \frac{1}{2}$ Eq. (31) reduces to the case quoted by Grinberg [8]. Grinberg's source was [9] which is unavailable to us.

5. Applications. Both Eq. (1) and Eq. (2) occur in a magneto-hydrodynamic duct flow problem in which the duct is rectangular and the walls parallel to the applied magnetic field are perfectly conducting, the other walls being insulators. Eq. (1) occurs in the form

$$\int_0^1 F(y)\,|x-y|^{-1/2}\,dy = 1$$

where $F(y)$ is the outward derivative of the fluid velocity at the perfectly conducting wall. Eq. (31) can be integrated with $f(x) = 1$. The inner integral is a fractional integral of the Riemann-Liouville type ([6, Vol. 2, p. 185]), while the outer integration can be easily carried out. The result is

$$F(x) = 2^{1/2}x^{-1/4}(1-x)^{-1/4}/2\pi.$$

Eq. (2) occurs in the form

$$\int_0^1 F(y)\,|x-y|^{-1/2}\,\mathrm{sgn}\,(x-y)\,dy = 2((1-x)^{1/2} - x^{1/2})$$

where $F(y)$ is the current density along the perfectly conducting wall. In this case Eq. (18) gives the solution

$$F(x) = -1 + c_2 x^{-3/4}(1 - x)^{-3/4}$$

It is considerably easier to show that (-1) is a particular solution of the integral equation than it is to carry out the integrations in Eq. (18). The details of this calculation can be found in [10].

It is hoped that the availability of these solutions in the literature will lead to further applications.

6. Generalization. The solutions of Eq. (1) and Eq. (2) can be used to construct solutions of a more general integral equation. Consider

$$\int_0^1 F(y)\, |x - y|^{m-\alpha} [\mathrm{sgn}\,(x - y)]^n\, dy = g(x) \tag{32}$$

where m is a positive integer, n is zero or one and $0 < \alpha < 1$. If we differentiate this m times, supposing $g(x)$ is sufficiently regular, we get

$$(1 - \alpha)_m \int_0^1 F(y)\, |x - y|^{-\alpha} [\mathrm{sgn}\,(x - y)]^{m+n}\, dy = g^m(x) \tag{33}$$

where

$$(a)_m = a(a + 1) \cdots (a + m - 1).$$

Equation (33) is the same as Eq. (1) or Eq. (2) depending on whether $m + n$ is even or odd.

Every solution of Eq. (32) is a solution of Eq. (33) but the converse is not true. In fact, solutions of Eq. (33) solve

$$\int_0^1 F(y)\, |x - y|^{m-\alpha} [\mathrm{sgn}\,(x - y)]^n\, dy = g(x) + P_{m-1}(x) \tag{34}$$

where P_{m-1} is a polynomial of degree $m - 1$. That is, if one takes a definite solution of Eq. (33) and substitutes it into the left-hand side of Eq. (32), one gets, not $g(x)$, but $g(x) + P_{m-1}$ with the coefficients in the polynomial completely determined by the solution of Eq. (33). A necessary and sufficient condition for Eq. (32) to be soluble is that conditions be put on $g(x)$ so that the coefficients in the polynomial be zero. These conditions can be derived by differentiating Eq. (34) j times with $j = 0, 1, 2 \cdots m - 1$ and then setting $x = 0$. This isolates the polynomial coefficients on the right which we then set equal to zero. The result of this calculation gives the conditions

$$(1 - \alpha + m - j)_j \int_0^1 F(y) y^{-\alpha+m-j} (-1)^{n+j}\, dy = g^j(0) \tag{35}$$

with $F(y)$ given by Eq. (18) if $m + n$ is odd or by Eq. (31) if $m + n$ is even, with $f(x) = g^m(x)/(1 - \alpha)_m$. This gives m conditions on the function $g(x)$ when $m + n$ is even and $m - 1$ conditions when $m + n$ is odd, since the constant c_1 which results from the nonuniqueness of Eq. (2) can be chosen to satisfy one of the conditions.

When the pertinent solutions are substituted into Eq. (35), the equation can be reduced considerably by changes in the order of integration, however, the result for arbitrary m is too clumsy to record here. We will give the two simplest cases, $m = 1$

and $n = 0$ or 1. In the first case, we have the equation

$$\int_0^1 F(y)\,|x - y|^{1-\alpha}\,dy = g(x). \tag{36}$$

The condition given by Eq. (35) can be reduced to

$$c_1\Gamma(1 - \alpha/2)\Gamma(\alpha/2) + g(0) - \frac{1}{h(\alpha)}\frac{\Gamma(1 - \alpha/2)}{\Gamma(\alpha/2)}\int_0^1 g(\beta)\beta^{-1+\alpha/2}(1 - \beta)^{-1+\alpha/2}\,d\beta = g(0)$$

which can be solved for c_1. The resulting solution of Eq. (36) is

$$h(\alpha)\Gamma^2(\alpha/2)F(x) = x^{-1+\alpha/2}(1 - x)^{-1+\alpha/2}\int_0^1 g(\beta)\beta^{-1+\alpha/2}(1 - \beta)^{-1+\alpha/2}\,d\beta$$

$$+ \frac{d}{dx}\,x^{\alpha/2}\int_x^1 \xi^{-\alpha}(\xi - x)^{-1+\alpha/2}\,d\xi\int_0^\xi \beta^{\alpha/2}(\xi - \beta)^{-1+\alpha/2}\,\frac{g'(\beta)}{1 - \alpha}\,d\beta$$

without restriction on $g(x)$.

In the second case, we have the equation

$$\int_0^1 F(y)\,|x - y|^{1-\alpha}\,\mathrm{sgn}\,(x - y)\,dy = g(x) \tag{37}$$

In this case Eq. (35) reduces to

$$\frac{1}{2}\,\frac{\Gamma(1/2 - \alpha/2)}{h'(\alpha)\Gamma(1/2 + \alpha/2)}\int_0^1 \beta^{-1/2+\alpha/2}(1 - \beta)^{-1/2+\alpha/2}\left\{g'(\beta) - (1 - \alpha)\,\frac{(g(\beta) - g(0))}{\beta}\right\}d\beta = g(0)$$

which is not satisfied by all functions, i.e. it is not satisfied by $g(x) = 1$. With this restriction on $g(x)$ the solution of Eq. (37) is given by Eq. (31) with $f(x) = g'(x)/1 - \alpha$.

Appendix (1)

Inverse Laplace Transform of $F^*(s)_+$. From Eq. (10)

$$h(\alpha)F^*(s)_+ = -ss^{-\alpha/2}Q(s)_+ \tag{A(1)}$$

where

$$Q(s)_+ = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f_1^*(w)_+(-w)^{-\alpha/2}\,\frac{dw}{w - s}.$$

Now

$$L^{-1}h(\alpha)F^*(s)_+ = -\frac{d}{dx}\,[L^{-1}s^{-\alpha/2}*L^{-1}Q(s)_+] \tag{A(2)}$$

where * indicates the convolution integral. The functions appearing in the convolution are

$$L^{-1}s^{-\alpha/2} = x^{-1+\alpha/2}/\Gamma(\alpha/2) \tag{A(3)}$$

and

$$
\begin{aligned}
L^{-1}Q(s)_+ &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f_1^*(w)_+(-w)^{-\alpha/2}e^{wx}H(x)\,dw \\
&= -H(x)L^{-1}[f_1^*(s)_+(-s)^{-\alpha/2}] \\
&= -H(x)[L^{-1}f_1^*(s)_+ * L^{-1}(-s)^{-\alpha/2}] \\
&= -H(x)/\Gamma(\alpha/2)\int_x^{\infty} f_1(\xi)(\xi - x)^{-1+\alpha/2}\,d\xi. \qquad \text{A(4)}
\end{aligned}
$$

Substituting A(3) and A(4) into the convolution integral A(2) gives

$$
h(\alpha)F(x) = \frac{1}{\Gamma(\alpha/2)^2}\frac{d}{dx}\int_0^x (x-\eta)^{-1+\alpha/2}\,d\eta \int_\eta^\infty f_1(\xi)(\xi-\eta)^{-1+\alpha/2}\,d\xi
$$

as was to be shown.

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