

A VARIATIONAL METHOD FOR THE LIMIT ANALYSIS OF ANISOTROPIC PLATES*

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1. Introduction. It has been shown by Mura and Lee [1] that the safety factor, the kinematically admissible multiplier and the statically admissible multiplier in the limit analysis of perfectly plastic solids are actually the extremum values of the same functional under different constraint conditions. Due to the complexity of these constraint conditions, however, the application of this functional to the calculation of the upper and lower bounds requires further simplification in the mathematical procedure. In this paper, some of the constraint conditions are removed and the upper and lower bounds are calculated respectively by means of two new functionals which have been extremized by variational procedure. The application of these functionals are illustrated by means of a simply supported circular orthotropic plate subjected to uniformly distributed load. The results obtained by the proposed method are compared with those obtained by the classical method.

2. Safety factor. The safety factor of an anisotropic plate for a given load p is the stationary value of the following functional [1]

$$F[\dot{w}, M_{ij}, R, m, \mu, \phi] = \int_A M_{ij}(-\dot{w}_{,ij}) dA - \int_S R\dot{w} dS - m \left(\int_A p\dot{w} dA - 1 \right) - \int_A \mu [f(M_{ij}) + \phi^2] dA \quad (1)$$

with the constraint condition

$$\mu \geq 0 \quad (2)$$

where

$$f(M_{ij}) = \frac{1}{2} C_{ijkl} M_{ij} M_{kl} - k^2 \quad (3)$$

and, j denotes differentiation with respect to x_j . In these equations, $\dot{w}_{,ij}$ denotes the curvature rate field associated with the velocity field \dot{w} , M_{ij} the bending moment field, k the yield parameter, A the domain, and S the boundary of the domain. Also C_{ijkl} are the material constants, ϕ is a point function and R , m and μ are Lagrangian multipliers.

The proof of the above statement is given as follows. Taking the variation of F leads to the following natural conditions:

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$$-\dot{w}_{,ii} = \mu \partial f / \partial M_{ii}, \quad \mu \geq 0, \quad \text{in } A, \quad (4)$$

$$M_{ii,ii} = -mp \quad \text{in } A, \quad (5)$$

$$M_{ij}n_j \delta \dot{w}_{,i} = 0 \quad \text{on } S, \quad (6)$$

$$M_{ij,ij}n_i = R \quad \text{on } S, \quad (7)$$

$$f(M_{ij}) + \phi^2 = 0 \quad \text{in } A, \quad (8)$$

$$\mu \phi = 0 \quad \text{in } A, \quad (9)$$

$$\dot{w} = 0 \quad \text{on } S, \quad (10)$$

$$\int_A p \dot{w} dA = 1. \quad (11)$$

Condition (4) is the plastic potential flow law, (5) to (7) are the equilibrium and boundary conditions, (8) and (9) define the admissible domain of the stress fields, i.e.,

$$f(M_{ij}) = 0, \quad \text{if } \mu > 0, \quad (12)$$

$$f(M_{ij}) \leq 0, \quad \text{if } \mu = 0,$$

and conditions (10) and (11) define the velocity field. It should be observed that the left side of (6) can be written in the form

$$M_n \partial(\delta \dot{w}) / \partial n + M_{ns} \partial(\delta \dot{w}) / \partial s = 0, \quad \text{on } S \quad (13)$$

where M_n and M_{ns} are the bending and twisting moments at the boundary respectively. Since the slope tangential to the boundary is zero from (10), (13) becomes

$$M_n \partial(\delta \dot{w}) / \partial n = 0, \quad \text{on } S. \quad (14)$$

When the plate is simply supported, (14) leads to

$$M_n = 0, \quad \text{on } S \quad (15)$$

since $\partial(\dot{w}) / \partial n$ is arbitrary in this case. If the plate is clamped, (14) becomes

$$\partial(\dot{w}) / \partial n = 0, \quad \text{on } S. \quad (16)$$

It is obvious that conditions (4) to (11) are the conditions for incipient plastic flow. Integrating the functional by parts in view of (4) to (11), it can be readily shown that the extremum value of F is equal to m . Therefore the safety factor s can be defined as the extremum value of F , i.e.,

$$\text{Ext } F = m = s \quad (17)$$

3. Lower bound theorem. A lower bound of the safety factor is

$$m^0 = m / \alpha^0 \quad (18)$$

where

$$\alpha^0 = \left[\frac{k^2}{\max \{ \frac{1}{2} C_{ijkl} M_{ij} M_{kl} \}} \right]^{1/2}. \quad (19)$$

In the above equations, m and M_{ij} are obtained by the direct method of variational

calculus using the following functional with arguments m, M_{ii}, μ and ϕ :

$$F^0 = m - \int_A \mu [f(M_{ii}) + \phi^2] dA \tag{20}$$

with constraint conditions (2) and (5) for clamped boundary, and conditions (2), (5) and (15) for simply supported boundary. The above functional is derived by integrating (1) by parts in view of (5), (6) and (7). The stress field M_{ii}/α^0 is statically admissible since

$$(M_{ii}/\alpha^0)_{,ii} = -(m/\alpha^0)p, \tag{21}$$

$$\frac{1}{2}C_{ijkl}(M_{ii}/\alpha^0)(M_{kl}/\alpha^0) \leq k^2, \tag{22}$$

$$M_{n}/\alpha^0 = 0, \text{ on } S \tag{23}$$

the last condition applicable only to simply supported boundary.

4. Upper bound theorem. An upper bound of the safety factor is

$$m^* = 2k^2 \int_A \mu \alpha^* dA \tag{24}$$

where α^* is a point function defined by

$$\frac{1}{2}C_{ijkl}M_{ii}M_{kl} = (\alpha^*)^2k^2. \tag{25}$$

In the above equations, μ and M_{ii} are obtained by the direct method of variational calculus using the following functional with arguments μ, M_{ii} and ϕ :

$$F^* = 2k^2 \int_A \mu dA + \int_A \mu [f(M_{ii}) - \phi^2] dA \tag{26}$$

with constraint conditions (2), (4), (10) and (11) for simply supported boundary, with the addition of condition (16) for clamped boundary. The above functional is derived from (1) in view of these constraint conditions and the fact that $M_{ii} \partial f / \partial M_{ii} = 2f + 2k^2$.

In order to prove that (24) is an upper bound, consider the following expression for the safety factor:

$$s = \int_A \bar{M}_{ii}(-\dot{w}_{,ii}) dA \tag{27}$$

where \bar{M}_{ii} is the true stress field and \dot{w} is a velocity field associated with (26) and the constraint conditions. In view of condition (4), the right-hand side in (27) becomes

$$\int_A \mu C_{ijkl}M_{kl}\bar{M}_{ii} dA \tag{28}$$

an application of Schwartz's inequality in Riemann space with fundamental tensors C_{ijkl} leads to

$$s \leq \int_A \mu (C_{ijkl}M_{ii}M_{kl})^{1/2} (C_{ijkl}\bar{M}_{ii}\bar{M}_{kl})^{1/2} dA. \tag{29}$$

In view of (25) and the fact that $\frac{1}{2}C_{ijkl}\bar{M}_{ii}\bar{M}_{kl} \leq k^2$, (20) becomes

$$s \leq 2k^2 \int_A \mu \alpha^* dA. \tag{30}$$

5. **Example.** Consider a simply supported circular orthotropic plate of radius a , subjected to a uniformly distributed load of intensity p , and assume that the principal directions of orthotropy are parallel to the axes $\theta = 0$ and $\theta = \pi/2$ (Fig. 1). The yield criterion suggested by Hill [2, 3] for a rigid plastic circular plate in terms of the generalized stresses is

$$f(M_{ij}) = C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2 + C_4 M_{r\theta}^2 + C_5 M_r M_{r\theta} + C_6 M_\theta M_{r\theta} - h^4 = 0 \quad (31)$$

in which

$$\begin{aligned} C_1 &= (G + H) \cos^4 \theta - \frac{1}{2} H \sin^2 2\theta + (F + H) \sin^4 \theta + \frac{1}{2} N \sin^2 2\theta, \\ C_2 &= -\frac{1}{2}(G + H) \sin^2 2\theta + 2H(\sin^4 \theta + \cos^4 \theta) - \frac{1}{2}(F + H) \sin^2 2\theta + N \sin^2 2\theta, \\ C_3 &= (G + H) \sin^4 \theta - \frac{1}{2} H \sin^2 2\theta + (F + H) \cos^4 \theta + \frac{1}{2} N \sin^2 2\theta, \\ C_4 &= (G + H) \sin^2 2\theta + 2H \sin^2 2\theta + (F + H) \sin^2 2\theta + 2N \cos^2 2\theta, \\ C_5 &= -2(G + H) \sin 2\theta \cos^2 \theta + H \sin 4\theta + 2(F + H) \sin 2\theta \sin^2 \theta + N \sin 4\theta, \\ C_6 &= -2(G + H) \sin 2\theta \sin^2 \theta + H \sin 4\theta + 2(F + H) \sin 2\theta \cos^2 \theta - N \sin 4\theta, \\ k^2 &= h^4. \end{aligned} \quad (32)$$

$$\begin{aligned} 2F &= \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}, \\ 2G &= \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}, \\ 2H &= \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}, \\ 2N &= \frac{1}{T^2}. \end{aligned} \quad (33)$$

In these equations, $2h$ is the plate thickness, X, Y, Z are the tensile yield stresses in the principal directions of orthotropy and T is the yield stress in shear with respect to these principal axes.

Lower bound. Assume a stress field

$$\begin{aligned} M_r &= \frac{1}{6} pm(a^2 - r^2), \\ M_\theta &= \frac{1}{6} pm a^2, \\ M_{r\theta} &= 0 \end{aligned} \quad (34)$$

satisfying the stress boundary condition and the equilibrium equation

$$\frac{d}{dr}(rM_r) - M_\theta = -\frac{1}{2} pmr^2. \quad (35)$$

The functional F^0 defined by (20) becomes in this case

$$F^0 = m - \int_0^{2\pi} \int_0^r \mu [C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2 - h^4 + \phi^2] r \, dr \, d\theta \quad (36)$$

For a family of orthotropic plates such that

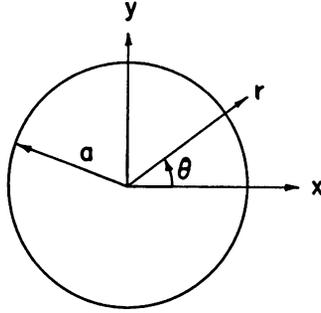


FIG. 1. Circular orthotropic plate.

$$F = G/\mu_1, \quad H = G, \quad N = 3G \quad (37)$$

in which μ_1 is a proportional constant, C_1 , C_2 and C_3 in (32) become

$$\begin{aligned} C_1 &= G \left[2 \cos^4 \theta + \sin^2 2\theta + \left(1 + \frac{1}{\mu_1} \right) \sin^4 \theta \right], \\ C_2 &= G \left[2(\sin^4 \theta + \cos^4 \theta) - \frac{1}{2} \left(\frac{1}{\mu_1} - 3 \right) \sin^2 2\theta \right], \\ C_3 &= G \left[2\sin^4 \theta + \sin^2 2\theta + \left(1 + \frac{1}{\mu_1} \right) \cos^4 \theta \right]. \end{aligned} \quad (38)$$

Substituting (34) and (38) in (36), assuming that $\mu = \mu_0 = \text{constant}$ and $\phi = \phi_0 = \text{constant}$ and integrating lead to

$$\begin{aligned} F^0 &= m - \mu_0 \left\{ \frac{\pi G a^6}{4} \left(\frac{pm}{6} \right)^2 \left[\frac{1}{6} \left(13 + \frac{3}{\mu_1} \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \left(18 - \frac{2}{\mu_1} \right) + \frac{1}{2} \left(13 + \frac{3}{\mu_1} \right) \right] - h^4 a^2 \pi + \phi_0^2 a^2 \pi \right\}. \end{aligned} \quad (39)$$

The values of the unknowns are determined by extremizing F^0 with respect to m , μ_0 and ϕ_0 , i.e.,

$$\partial F^0 / \partial m = 0, \quad \partial F^0 / \partial \mu_0 = 0 \quad \text{and} \quad \partial F^0 / \partial \phi_0 = 0.$$

Solving these equations yields

$$\begin{aligned} m &= \frac{12(2)^{1/2}}{\left[\frac{25}{6} + \frac{5}{2} \frac{1}{\mu_1} \right]^{1/2}} \frac{M_0}{pa^2}, \\ \mu_0 &= \frac{6(2)^{1/2}}{\left[\frac{25}{6} + \frac{5}{2} \frac{1}{\mu_1} \right]^{1/2}} \frac{M_0}{\pi pa^4 h^4} > 0, \\ \phi_0 &= 0 \end{aligned} \quad (40)$$

in which

$$M_0 = \frac{h^2}{(2G)^{1/2}} = Xh^2. \quad (41)$$

The constant α^0 defined by (19) takes the form, in this case,

$$\alpha^0 = \left[\frac{h^4}{\max \{C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2\}} \right]^{1/2}. \tag{42}$$

For a given value of μ_1 , m is given by (40), $\max \{\frac{1}{2} C_{i,jk} M_{i,j} M_{k,l}\}$ is calculated with the aid of (34), (38) and (40), and m^0 by (18) and (42). The values of m^0 are plotted in Fig. 2 against arguments of μ_1 in terms of M_0/pa^2 .

Upper bound. Assume the velocity field

$$\dot{w} = A_1(a - r) \tag{43}$$

in which A_1 is an arbitrary constant, satisfying (10), the boundary condition at $r = a$. Substituting (43) in (11) and solving for A_1 yields

$$A_1 = \frac{3}{\pi p a^3}. \tag{44}$$

The condition (4) becomes in this case

$$\begin{aligned} -\dot{w}_{,rr} &= \mu(2C_1 M_r - C_2 M_\theta), \\ -\frac{\dot{w}_{,r}}{r} &= \mu(2C_3 M_\theta - C_2 M_r). \end{aligned}$$

Solving simultaneously for M_r and M_θ with the aid of (43) and (44) and assuming

$$\mu = \frac{A_2}{r} \tag{45}$$

in which A_2 is a constant lead to

$$\begin{aligned} M_r &= \frac{C_2}{4C_1 C_3 - C_2^2} \frac{3}{\pi p a^3 A_2}, \\ M_\theta &= \frac{2C_1}{4C_1 C_3 - C_2^2} \frac{3}{\pi p a^3 A_2}. \end{aligned} \tag{46}$$

The functional F^* defined by (26) takes the form

$$\begin{aligned} F^* &= 2h^4 \int_0^{2\pi} \int_0^a \mu r \, dr \, d\theta \\ &+ \int_0^{2\pi} \int_0^a \mu [C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2 - h^4 - \phi^2] r \, dr \, d\theta. \end{aligned} \tag{47}$$

Substituting (45) and (46) in (47), assuming $\phi = 0$ and integrating yield

$$F^* = 4\pi a h^4 A_2 + \frac{9a N}{A_2 (\pi p a^3)^2} - 2\pi a h^4 A_2 \tag{48}$$

in which

$$N = \int_0^{2\pi} \frac{C_1}{4C_1 C_3 - C_2^2} \, d\theta. \tag{49}$$

Extremizing F^* with respect to A_2 , i.e.,

$$\partial F^* / \partial A_2 = 0$$

and solving for A_2 yield

$$A_2 = \frac{3}{\pi p a^3 h^2} \left(\frac{N}{2\pi} \right)^{1/2} \tag{50}$$

It should be observed that (45) and (50) satisfy constraint condition (2). Condition (25) in this case takes the form

$$C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2 = (\alpha^*)^2 h^4.$$

Solving for α^* yield

$$\alpha^* = \frac{(C_1 M_r^2 - C_2 M_r M_\theta + C_3 M_\theta^2)^{1/2}}{h^2}. \tag{51}$$

The upper bound of the safety factor m^* defined by (24) becomes

$$m^* = 2h^4 \int_0^{2\pi} \int_0^a \mu \alpha^* r \, dr \, d\theta.$$

Substituting (45) and (51) with the aid of (46) and (50) and integrating yield the upper bound of the safety factor,

$$m^* = \frac{6h^2}{\pi p a^2} \int_0^{2\pi} \left(\frac{C_1}{4C_1 C_3 - C_2^2} \right)^{1/2} d\theta. \tag{52}$$

The values of m^* are plotted in Fig. 2 against arguments of μ_1 in terms of M_0/pa^2 .

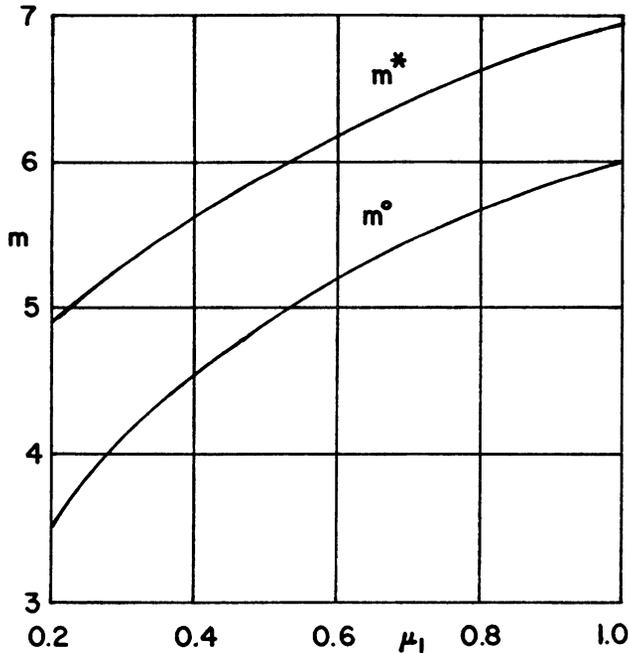


FIG. 2. Values of multipliers plotted against arguments of μ_1 .

6. Conclusion. Distinct from the extended theorems of limit analysis presented recently by Mura, Rimawi and Lee [4] in which the yield condition is replaced by a weighted integral mean of the yield function, the present method satisfies all the constraint conditions of the classical theorems. It is of interest to note that the lower and upper bounds obtained by the proposed method are in this instance the same as those obtained by the classical method [5]. In the proposed method, however, the two functionals, F^0 and F^* provide the means by which both the upper and the lower bounds can be obtained by variational procedure. Furthermore it is recalled that, in the lower bound analysis of the classical method, the point at which $\max [\frac{1}{2}C_{ijkl}M_{ij}M_{kl}]$ occurs has to be guessed a priori in order to satisfy the yield condition $f \leq 0$. Should the wrong assumption be made, the calculation has to be repeated all over again. In the proposed method, on the other hand, this point is located at the end of the procedure after all the constraint conditions have been satisfied, hence no guessing is necessary in this respect. Last, but not the least, it should be mentioned that no functional, such as F^0 , is up to now available for lower bound analysis in the classical method of limit analysis.

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