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NONLINEAR OSCILLATIONS IN A DISTRIBUTED NETWORK*

BY

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Abstract. The oscillations of small amplitude in a lossless transmission line terminated with a nonlinear circuit are studied by perturbation theory. The equations describing this system are reduced to a difference-differential equation with one delay. A general procedure is given for equations of this type for finding the expansion of the oscillation to any order in terms of the coefficient of the fundamental frequency. The frequency-amplitude relations are obtained to second order and compared with results found on the computer. Both the autonomous and forced cases are studied. It is indicated in the forced case that the frequency-amplitude relation gives approximately the range of "locking in".

Introduction. Historically, although the basic mathematical ideas can be traced back to Poincaré, the work of van der Pol introduced one of the main physical problems in the study of nonlinear oscillations. This work originated out of the study of the differential equation describing a triode oscillator and has subsequently become known as the van der Pol equation. Today, the methods introduced by van der Pol, Duffing, Bogoliubov, and Mitropolski (see [1], [2], and [3]), to name a few, are familiar tools to many electrical engineers for studying the nature of nonlinear oscillations in electrical networks.

Recently, there has been great interest in the study of nonlumped or distributed networks. Since these networks can exhibit oscillatory behavior, it is natural to ask for some mathematical tools to study this behavior. It is the purpose of this paper to give some methods for this purpose. The methods used are conceptually similar to those existing for ordinary differential equations.

In this paper we study a particular circuit consisting of a lossless transmission line terminated by lumped circuits, one of which is nonlinear. This circuit is described in Section 1 where the equations are derived also. In Section 2 these equations are reduced to a single difference-differential equation of neutral type while in Section 3 the experimental behavior (obtained by numerical integration) of this circuit is discussed. The linear part of the equation is analyzed in Section 4 through the analysis

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of the associated characteristic equation. In Section 5 the equations are put in the canonical form

$$L_0(u) = F(u, \epsilon), \tag{1}$$

where $L_0(\cdot)$ is a linear difference-differential operator which has one pair of characteristic roots $\pm i$ with the remainder lying in the left half plane. $F(u, \epsilon)$ is a nonlinear difference-differential operator which is $O(\epsilon)$ as $\epsilon \to 0$. The periodic solution of (1) and its period is then determined by an expansion in ϵ. This procedure for obtaining the periodic solution is given for the general equation of the form (1). In Section 6 this procedure is applied to the circuit in question to obtain specific results which are compared to results obtained numerically. Section 7 contains the discussion of the nonautonomous case where the equations are of the form

$$L_0(u) = F(u, t, \epsilon)$$

and $F(u, t, \epsilon)$ is periodic in t of period $2\pi/\omega$. This case corresponds to the case of a periodic current source in the circuit. Results obtained here are also compared to numerical results.

1. The network and network equations. The circuit being considered is shown in Fig. 1 where the current through the resistor in the direction shown is given by $f(v_1)$,

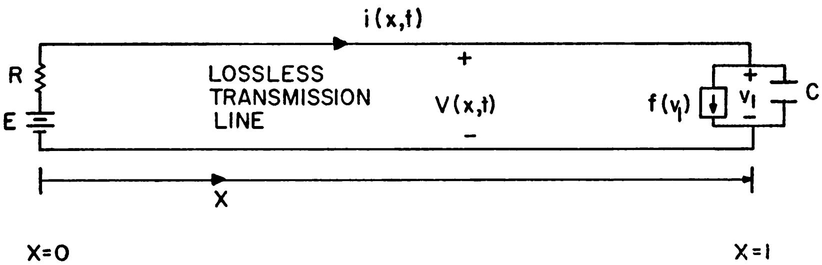

FIG. 1.

a nonlinear function of the voltage v_1. The length of the line is normalized to be one. If L_s, C_s are the specific inductance and capacitance in the line, then the equations for $0 \leq x \leq 1$ are*

$$\begin{aligned} L_s i_t &= -v_x , \\ C_s v_t &= -i_x . \end{aligned} \tag{1.1}$$

Here $i(x, t)$, $v(x, t)$ are, respectively, the current in the line and the voltage across the line at the point x. These equations together with the boundary conditions

$$\begin{aligned} E - v(0, t) - Ri(0, t) &= 0, \qquad & x = 0, \\ C\, dv(1, t)/dt &= i(1, t) - f(v(1, t)), \qquad & x = 1, \end{aligned} \tag{1.2}$$

completely characterize the system.

Under equilibrium conditions, $i_x = v_x = 0$. Thus, $i_0 = i_1$ and $v_0 = v_1$ where i_0, etc.

*v_x, v_t, etc. denote $\partial v/\partial x$, $\partial v/\partial t$, respectively, etc.

denotes $i(x, t)$ evaluated at $x = 0$, etc. Therefore, from (1.2) we have the equilibrium equations

$$E - v_1 - Ri_1 = 0,$$
$$i_1 = f(v_1).$$

The solution of these equations can be found graphically as illustrated in Fig. 2. Here,

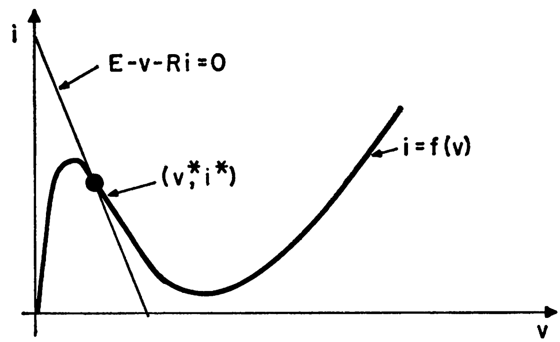

FIG. 2.

we have chosen E and R so that there is only one intersection (v^*, i^*) which is at the point of maximum negative slope. By changing variables, the equilibrium can be shifted from (v^*, i^*) to $(0, 0)$. The equations are then

$$\begin{aligned} L_s i_t &= -v_x , \\ C_s v_t &= -i_x , \\ 0 &= v_0 + Ri_0 , \\ Cv_{1t} &= i_1 - g(v_1). \end{aligned} \tag{1.3}$$

For the purpose of obtaining specific results, we assume that

$$g(v_1) = -gv_1 + v_1^3 .$$

2. Reduction to a nonlinear difference-differential equation. The system (1.3) of partial differential equations with boundary conditions can be reduced to a single nonlinear difference-differential equation. To do this we use the fact that any solution of the wave equation (1.1) can be expressed as a linear combination of two waves, one traveling to the right, $\phi(x - st)$, and one to the left, $\psi(x + st)$, where s is the velocity of propagation. In particular,

$$v(x, t) = (1/2)(\phi(x - st) + \psi(x + st)),$$
$$i(x, t) = (1/2z)(\phi(x - st) - \psi(x + st)),$$

where $s = 1/(L_s C_s)^{1/2}$ and $z = (L_s/C_s)^{1/2}$. Since $\phi(x - st)$ is a wave traveling to the right, then

$$\phi_1(t) = \phi_0(t - h/2) = (v_0 + zi_0)(t - h/2),$$

where $h = 2(L_s C_s)^{1/2}$. Similarly,

$$\psi_1(t - h) = \psi_0(t - h/2) = (v_0 - zi_0)(t - h/2).$$

Using the boundary condition $v_0 = -Ri_0$, we obtain

$$\phi_1(t) = K\psi_1(t - h), \tag{2.1}$$

where

$$K = (R - z)/(R + z).$$

K can be interpreted as the reflection coefficient of the left-hand terminal. Using (2.1), the right-hand boundary condition yields

$$C(v_1'(t) + Kv_1'(t - h)) + (1/z - g)v_1(t) - K(1/z + g)v_1(t - h) = -v_1^3(t) - Kv_1^3(t - h). \tag{2.2}$$

We write this symbolically as

$$L(v_1) = G(v_1),$$

where L is a linear difference-differential operator and G is a nonlinear operator. Eq. (2.2) is a single nonlinear difference-differential equation with one time delay h and is equivalent to the system of Eqs. (1.3). Clearly, by knowing $v_1(t)$, $v_1'(t)$, the behavior of $v(x, t)$ or $i(x, t)$ is easily computed by using the relations derived above. Although the analysis of this problem could have been done in the x, t variables for the system (1.1)–(1.2), the reduction to (2.2) simplifies the details and makes the numerical integration easier.

3. Experimental behavior—numerical results. We shall be discussing the behavior of the circuit shown in Fig. 1 under conditions which can cause oscillations:

$$R < z < 1/g.$$

In this case there exists a value $R_0 < z$ such that if $R > R_0$, then the equilibrium state (0, 0) is stable and no oscillations occur. However, as R is decreased past R_0, the equilibrium state becomes unstable and a small oscillation appears with some frequency ω_0. Further decreasing R causes the amplitude and frequency of this oscillation to increase.

This behavior can be demonstrated by integrating Eq. (2.2) numerically. We note that one advantage in reducing the equations to a difference-differential equation is that this equation can be solved numerically essentially like an ordinary differential equation. The periodic solution is obtained by simply computing long enough, since any initial data decay exponentially to the periodic solution.

This was done where the following numerical values were used for the circuit parameters

$$z = 25 \text{ ohms}, \qquad C = 10 \text{ p.f.}, \qquad h = 2 \text{ n.s.}, \qquad g = .01 \text{ mho}. \tag{3.1}$$

The following table shows the amplitude A (maximum positive value of v_1) and frequency ω for different values of R.

Table 1 (see also Fig. 5) indicates that there is a nonlinear relation between frequency and amplitude much as there is in certain mechanical and electrical networks also exhibiting self-sustained oscillations (e.g., see [2]).

4. The characteristic equation. As in ordinary differential equations it is important to analyze the characteristic equation of the system linearized around the equilibrium

TABLE 1

R ohms	A volts	ω rad./n.s.
5.4	.0066	1.2509
5.3	.0190	1.2516
5.2	.0329	1.2532
5.0	.0392	1.2542
4.5	.0517	1.257
3.5	.0706	1.264

solution. In our case the linearized equation is

$$L(v) \equiv C[v'(t) + Kv'(t - h)] + (1/z - g)v(t) - K(1/z + g)v(t - h) = 0, \qquad (4.1)$$

and the characteristic equation (obtained by substituting $v = e^{st}$, s complex, into $L(v) = 0$) is

$$q(s) \equiv C[s + Kse^{-sh}] + (1/z - g) - K(1/z + g)e^{-sh} = 0. \qquad (4.2)$$

$q(s)$ is an exponential polynomial of the type studied by Pontryagin [5] and has an infinite number of roots. It is known (see [4]) that any solution $v(t)$ of (4.1) can be written as

$$v(t) = \sum_{r=1}^{\infty} \exp(s_r t) p_r(t),$$

where the $\{s_r\}$ are the roots of $q(s) = 0$ and $p_r(t)$ is a polynomial in t of degree less than the multiplicity of the root s_r. Furthermore, if all the roots satisfy $\text{Re}(s_r) \leq \gamma < 0$, then the origin is asymptotically stable. This statement also holds locally for the nonlinear equation (2.2) (see [9]).

An oscillation appears when a root of the characteristic equation crosses the imaginary axis. We, therefore, look for conditions on K under which an imaginary root $s = i\omega$ exists. Substituting into (4.2), we have

$$q(i\omega) = C[i\omega + Ki\omega \exp(-i\omega h)] + (1/z - g) - K(1/z + g) \exp(-i\omega h) = 0, \qquad (4.3)$$

which reduces to the following two equations:

$$\omega^2 = \frac{K^2(1/z + g)^2 - (1/z - g)^2}{C^2(1 - K^2)}, \qquad (4.4)$$

$$\arg(K) + \arg(-1/z - g + i\omega C) - \omega H = \arg(g - 1/z - i\omega C) + 2n\pi. \qquad (4.5)$$

Equation (4.5) reduces to

$$\tan \omega h = \frac{2C\omega}{z(-1/z^2 + g^2 + C^2\omega^2)}, \qquad (4.6)$$

which can be solved graphically as shown in Fig. 3. The intersections marked with an x would be roots if K were positive. The values ω_0, ω_1, $\cdots$ are roots of (4.3) if they also satisfy Eq. (4.4). For small values of K, there will be no root because then

$$K^2(1/z + g)^2 - (1/z - g)^2 < 0, \qquad 1 - K^2 > 0$$

which, by (4.4), implies that $\omega^2 < 0$. However, if K^2 is increased, there is a value K_0^2 such that ω_0 satisfies (4.4). Thus, for $K = K_0$, $\pm i\omega_0$ is a root of the characteristic equation

It is easy to show that all other characteristic roots have Re $(s_r) < 0$. It can also be shown that all roots lie to the right of the line Re $(s) = (1/h) \log |K|$ and as Im $(s) \to \pm\infty$, Re $(s) \to (1/h) \log |K|$. Thus, increasing K^2 shifts the whole spectrum to the right in some manner. This is illustrated in Fig. 4. We see, therefore, that K_0 is the value of K where a pair of characteristic roots crosses the imaginary axis for the first time and oscillation begins. We also see that the frequency of the initial oscillation is ω_0.

Finally, we point out that there is only one pair of roots that crosses the imaginary axis at this value K_0 (called the critical value of K) and that these roots are not multiple roots.

5. Perturbation expansion; the autonomous case. By making the change of variables $v = \beta x$ and $\omega t = \tau$, the equation (2.2) becomes

$$C\omega(x' + Kx'(\tau - \omega h)) + (1/z - g)x(\tau) - K(1/z + g)x(\tau - \omega h) = -\epsilon x^3(\tau) - K\epsilon x^3(\tau - \omega h) \tag{5.1}$$

where $\epsilon = \beta^2$ and $x' = dx/d\tau$. We consider ϵ small, which means that only oscillations

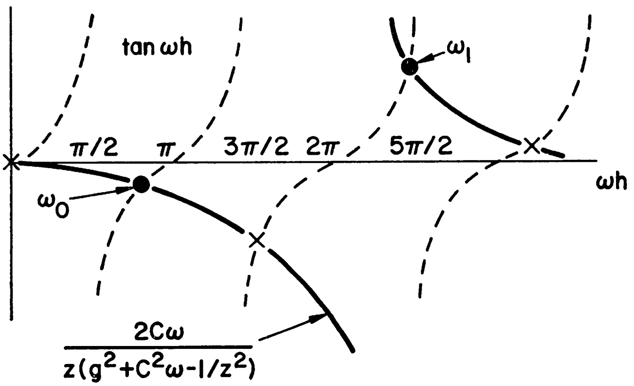

Fig. 3.

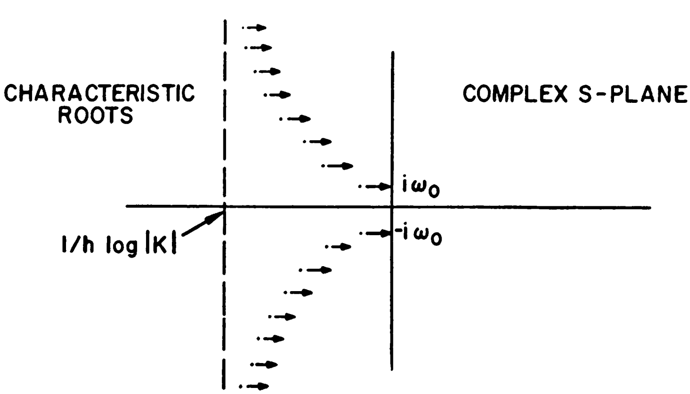

Fig. 4

with small amplitudes are being considered. Rewriting Eq. (5.1) in terms of ω_0 and K_0, we have

$$C\omega_0(x'(\tau) + K_0x'(\tau - \omega_0h)) + (1/z - g)x(\tau) - K_0(1/z + g)x(\tau - \omega_0h)$$
$$= -C(\omega - \omega_0)x'(\tau) - C[\omega Kx'(\tau - \omega h) - \omega_0K_0x'(\tau - \omega_0h)]$$
$$+ (1/z + g)[Kx(\tau - \omega h) - K_0x(\tau - \omega_0h)] - \epsilon x^3(\tau) - K\epsilon x^3(\tau - \omega h) \qquad (5.2)$$

The left-hand side coincides with the linear difference-differential operator $L(\cdot)$ defined by Eq. (4.1) if $K = K_0$, $\omega = \omega_0$, and $\tau = \omega_0 t$. We denote this operator by $L_0(\cdot)$ and its characteristic equation by $q_0(s) = 0$. According to the discussion in Section 4, this operator has characteristic roots $\pm i$; i.e., L_0 has a null space spanned by exp $(\pm i\tau)$.

We now look for a periodic solution of Eq. (5.2) in the form

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots,$$
$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots,$$
$$K = K_0 + \epsilon K_1 + \epsilon^2 K_2 + \cdots.$$

Note that ω and K are considered as unknowns rather than ω and ϵ.

Equation (5.2) can be written as

$$L_0(x) = F(x, \epsilon), \qquad (5.3)$$

where $F(\cdot, \epsilon)$ is a nonlinear difference-differential operator of order $O(\epsilon)$. Expanding and equating coefficients of ϵ, we obtain the following infinite set of equations for $x_0, x_1, \cdots$

$$\begin{aligned} L_0(x_0) &= 0, \\ L_0(x_1) &= \frac{dF}{d\epsilon}(x_0, \epsilon)\bigg|_{\epsilon=0}, \\ L_0(x_2) &= \frac{1}{2!}\frac{d^2F}{d\epsilon^2}(x_0 + \epsilon x_1, \epsilon)\bigg|_{\epsilon=0}, \\ &\vdots \\ L_0(x_n) &= \frac{1}{n!}\frac{d^nF}{d\epsilon^n}(x_0 + \cdots + \epsilon^{n-1}x_{n-1}, \epsilon)\bigg|_{\epsilon=0}. \\ &\vdots \end{aligned} \qquad (5.4)$$

Thus, if $x = x_0 + \epsilon x_1 + \cdots$ converges and $x_0, x_1, \cdots$ are periodic solutions of (5.4), then x is a periodic solution of Eq. (5.3) with period ω in t. However, it is not to be expected that this series converges and, in general, it is only an asymptotic expansion. The existence of a periodic solution of problems of this type is proved in another paper [6] by an iteration method rather than an expansion method.

We must show that each of the equations of (5.4) has a 2π-periodic solution. According to the Fredholm alternative, it is necessary that the right-hand sides of (5.4) be orthogonal to the periodic solutions of the adjoint equation of L_0. The adjoint equation is (see [4])

$$L^*(u) = -C\omega_0(u'(\tau) + K_0u'(\tau + \omega_0h)) + (1/z - g)u(\tau) - K_0(1/z + g)u(\tau + \omega_0h) = 0,$$

and its characteristic equation is

$$q_0^*(s) = -C\omega_0(s + K_0 s \exp(s\omega_0 h)) + (1/z - g) - K_0(1/z + g)\exp(s\omega_0 h) = 0.$$

Obviously, by taking complex conjugates, purely imaginary roots of $q_0(s) = 0$ are roots of $q_0^*(s) = 0$ and have the same multiplicity, and conversely. Thus, $\pm i$ are the only purely imaginary roots of $q_0^*(s) = 0$, and, therefore, $\sin\tau$, $\cos\tau$ span the space of periodic solutions of the adjoint equation. We, therefore, require that

$$\begin{aligned} 0 &= \int_0^{2\pi} \left. \frac{d^n F(x_0 + \cdots + \epsilon^{n-1} x_{n-1}, \epsilon)}{d\epsilon^n} \right|_{\epsilon=0} \sin\tau\, d\tau \\ &= \int_0^{2\pi} \left. \frac{d^n F(x_0 + \cdots + \epsilon^{n-1} x_{n-1}, \epsilon)}{d\epsilon^n} \right|_{\epsilon=0} \cos\tau\, d\tau \end{aligned} \tag{5.5}$$

for all n. These conditions are known generally as the bifurcation equations or secular equations and, as we shall see in Section 6, they are the relations required for determining ω_n and K_n.

Given the relations (5.5), we must show that (5.4) can be solved for periodic x_n. For this purpose we expand each right-hand side as a Fourier series. In the nth equation we have

$$L_0(x_n) = \sum_{n=2}^{\infty} a_n e^{in\tau} + \bar{a}_n e^{-in\tau}, \tag{5.6}$$

and we look for a solution

$$x_n = \sum_{n=2}^{\infty} c_n e^{in\tau} + \bar{c}_n e^{-in\tau}. \tag{5.7}$$

Note that the terms with frequency unity have been arbitrarily excluded from x_n.

Eq. (5.6) can be solved term by term; i.e., c_n can be chosen such that

$$L_0(c_n e^{in\tau}) = a_n e^{in\tau},$$

for all $n \geq 2$. This reduces to

$$c_n q_0(in) = a_n,$$

which has the solution

$$c_n = a_n / q_0(in) \tag{5.8}$$

since $q_0(in) \neq 0$ for $n \geq 2$.

Thus, Eqs. (5.4) can be solved for periodic solutions of period 2π. This discussion applied generally to scalar difference-differential equations of the form

$$L(u) = F(u, \epsilon),$$

where $L(\cdot)$ is a linear difference-differential operator with a single pair of imaginary roots, and $F(\cdot, \epsilon)$ is a nonlinear difference-differential operator of order ϵ.

6. Computations for the autonomous case; the frequency-amplitude relation. To illustrate the procedure described in Section 5, we solve the first two equations of (5.4). The bifurcation equations (5.5) for $n = 1$ will determine, in particular, ω_1, which with ω_0 determines an approximate frequency-amplitude relation $\omega = \omega_0 + \epsilon\omega_1$ where $\epsilon^{1/2}$ is the amplitude.

The first equation of (5.4)

$$L_0(x_0) = 0$$

has the solution $x_0(\tau) = \cos\tau$ where we have eliminated the $\sin\tau$ term using the arbitrariness of the origin of time in the autonomous case. The coefficient of $\cos\tau$ is taken as one since ϵ is still unspecified. With this value for x_0, the second equation of (5.4) becomes

$$\begin{aligned} L_0(x_1) &= C\omega_1 \sin\tau + C\omega_1 K_0 \sin(\tau - \omega_0 h) + C\omega_0 K_1 \sin(\tau - \omega_0 h) \\ &+ K_1(1/z + g)\cos(\tau - \omega_0 h) - \tfrac{3}{4}\cos\tau - \tfrac{1}{4}\cos 3\tau - \tfrac{3}{4}K_0\cos(\tau - \omega_0 h) \\ &- \tfrac{1}{4}K_0 \cos 3(\tau - \omega_0 h) - C\omega_0\omega_1 h K_0 \cos(\tau - \omega_0 h) + K_0(1/z + g)\omega_1 h \sin(\tau - \omega_0 h). \end{aligned} \quad (6.1)$$

The bifurcation equations (5.5) lead to the relation

$$\begin{aligned} (i\omega_1 C + \tfrac{3}{4})(1 + K_0 \exp(-i\omega_0 h)) + i\omega_1 h K_0 \exp(-i\omega_0 h)(1/z + g - i\omega_0 C) \\ = K_1 \exp(-i\omega_0 h)(-i\omega_0 C + 1/z + g) \end{aligned} \quad (6.2)$$

from which ω_1, K_1 can be determined.

We are then left with the following equation for $x_1(\tau)$:

$$\begin{aligned} L_0(x_1) &= -\tfrac{1}{4}(\cos 3\tau + K_0 \cos 3(\tau - \omega_0 h)) \\ &= -\tfrac{1}{8}[(1 + K_0 \exp(-i3\omega_0 h))\exp(i3\tau) + (1 + K_0 \exp(i3\omega_0 h))\exp(-i3\tau)]. \end{aligned}$$

From (5.7) and (5.8) we have

$$x_1(\tau) = c_3 \exp(i3\tau) + \bar{c}_3 \exp(-i3\tau),$$

where

$$c_3 = \frac{-\tfrac{1}{8}(1 + K_0 \exp(-i3\omega_0 h))}{i3C(1 + K_0 \exp(-i3\omega_0 h)) + (1/z - g) - K_0(1/z + g)\exp(-i3\omega_0 h)} \quad (6.3)$$

which completely determines $x_1(\tau)$.

To obtain the frequency-amplitude relation we combine Eqs. (6.2) and the following equation for ω_0, K_0:

$$i\omega_0 C[1 + K_0 \exp(-i\omega_0 h)] + (1/z - g) - K_0(1/z + g)\exp(-i\omega_0 h) = 0. \quad (6.4)$$

Multiplying (6.2) by ϵ adding (6.4), we have, within an error $O(\epsilon^2)$,

$$\hat{q}(i\omega) \equiv (i\omega C + 1/z - g + 3\epsilon/4) + \exp(-i\omega h)K(i\omega C - 1/z - g + 3\epsilon/4) = 0. \quad (6.5)$$

Eliminating K we obtain (since $K < 0$)

$$\tan\omega h = \frac{2C\omega}{z(-1/z^2 + (g - 3\epsilon/4)^2 + C^2\omega^2)} \quad (6.6)$$

which is the frequency-amplitude relation with error $O(\epsilon^2)$. Note that Eq. (6.5) is identical to $q(i\omega) = 0$ if g is replaced by $(g - 3\epsilon/4)$. This relation could also have been obtained by proceeding in a manner analogous to Duffing's iteration method (see [2]) for ordinary differential equations.

To compare the accuracy of the frequency-amplitude relation (6.5) we use Eq. (6.6) to determine ω for the set of parameters (3.1) for various values of $\beta = \epsilon^{1/2}$. These are compared in Fig. 5 with the values given in Table 1, Section 3. As expected there is good agreement for small values of amplitude β.

7. The frequency-amplitude relation for the nonautonomous case. We consider the network shown in Fig. 6 with a periodic current source of value $I_1 = I_0 \cos(\omega t - \theta)$ where θ is an undetermined phase. Proceeding as in Sections 1 and 2, the equations for this network can be reduced to

$$C[v_1'(t) + Kv_1'(t-h)] + (1/z - g)v_1(t) - K(1/z + g)v_1(t-h)$$
$$= -v_1^3(t) - Kv_1^3(t-h) + I_0 \cos(\omega t - \theta) + KI_0 \cos(\omega t - \theta - h) \qquad (7.1)$$

or

$$L(v_1) = G(v_1, t).$$

Again as in Section 5 we change variables $v_1 = \beta x$, $\tau = \omega t$, where $\epsilon = \beta^2$ is considered small. This leads to the equation

$$L_0(x) = F(x, \epsilon) + (I_0/\beta)[\cos(\tau - \theta) + K\cos(\tau - \theta - \omega h)], \qquad (7.2)$$

where L_0 and F are defined in Section 5. At this point we assume that $I_0/\beta = O(\epsilon) = \epsilon I$, which will give a forcing term that is compatible with our assumption that the amplitude of the response is small.

Proceeding as if K, ω are unknown and ϵ, I and θ are known, we look for a solution

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \cdots,$$
$$\omega = \omega_0 + \epsilon\omega_1 + \cdots,$$
$$K = K_0 + \epsilon K_1 + \cdots,$$

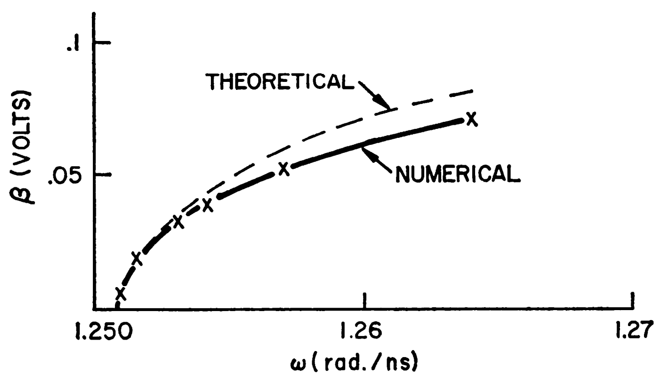

Fig. 5.

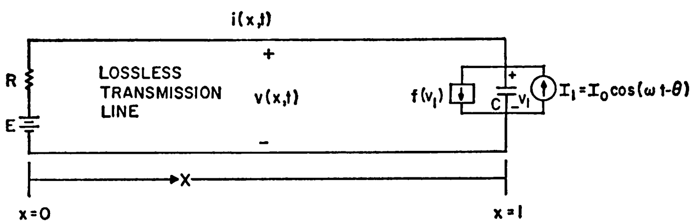

Fig. 6.

where $x_0(\tau) = \cos \tau$, ω_0 and K_0 are the same as in Section 4, and $x_n(\tau)$ is periodic of period 2π. This leads to the infinite set of equations given by (5.4) with F replaced by

$$F(x, \epsilon) + \epsilon I[\cos (\tau - \theta) + K \cos (\tau - \theta - \omega h)]. \tag{7.3}$$

Also the bifurcation equations (5.5) agree except for the above replacement. Finally, the method of determining $x_1(\tau)$, $x_2(\tau)$, etc., is the same as in Section 5.

Thus, the only difference in the nonautonomous case is that $F(x, \epsilon)$ is replaced by (7.3). We shall see the effect of this by examining the bifurcation equations. For the network under consideration, we have for the second equation of (5.4)

$$\begin{aligned} L_0(x_1) = {} & C_1\omega_1 \sin \tau + C_1\omega_1 K_0 \sin (\tau - \omega_0 h) + C\omega_0 K_1 \sin (\tau - \omega_0 h) \\ & + K_1(1/z + g) \cos (\tau - \omega_0 h) - \tfrac{3}{4} \cos \tau - \tfrac{1}{4} \cos 3\tau - \tfrac{3}{4}K_0 \cos (\tau - \omega_0 h) \\ & - \tfrac{1}{4}K_0 \cos 3(\tau - \omega_0 h) - C\omega_0\omega_1 h K_0 \cos (\tau - \omega_0 h) \\ & + K_0(1/z + g)\omega_1 h \sin (\tau - \omega_0 h) + I[\cos (\tau - \theta) + K_0 \cos (\tau - \theta - \omega_0 h)]. \end{aligned}$$

The bifurcation equations lead to the relation

$$\begin{aligned} (i\omega_1 C + \tfrac{3}{4})(1 + K_0 \exp (-i\omega_0 h)) + i\omega_1 h K_0 \exp (-i\omega_0 h)(1/z + g - i\omega_0 C) \\ = K_1 \exp (-i\omega_0 h)(1/z + g - i\omega_0 C) + Ie^{-i\theta}(1 + K_0 \exp (-i\omega_0 h)). \end{aligned}$$

Multiplying by ϵ and adding

$$i\omega_0 C(1 + K_0 \exp (-i\omega_0 h)) + (1/z - g) - K_0(1/z + g) \exp (-i\omega_0 h) = 0$$

yields to within error $O(\epsilon^2)$

$$(i\omega C + 1/z - g + \tfrac{3}{4}\beta^2 - (I_0/\beta)e^{-i\theta}) + Ke^{-i\omega h}(i\omega C - 1/z - g + \tfrac{3}{4}\beta^2 - (I_0/\beta)e^{-i\theta}) = 0.$$

Now since, in fact, ω and K are known, we can look at the above relation as an equation determining β and θ given ω, K and I_0. Eliminating θ gives the following cubic equation for β^2,

$$\begin{aligned} (1 + K^2 + 2K \cos \omega h)[(g - \tfrac{3}{4}\beta^2)^2 + \omega^2 C^2 - I_0^2/\beta^2] \\ + (1/z^2)(1 + K^2 - 2 \cos \omega h) - (1 - K^2)(2/z)(g - \tfrac{3}{4}\beta^2) = 0 \end{aligned} \tag{7.4}$$

which is the frequency-amplitude relation.

As might be expected, this relation for β^2 as a function of ω can be multivalued. This is indeed the case for some values of the parameters. Figure 7 shows the results for the parameter set

$$z = 25 \text{ ohms}, \quad R = 5 \text{ ohms}, \quad g = .0125 \text{ mho}, \quad C = 10^{-11} \text{ p.f.}, \quad h = 2 \text{ n.s.}$$

for different values of I_0. Note that the two parts of the curve merge as I_0 is increased. Such curves are what might be expected in a forced nonlinear ordinary differential equation with damping. An example is the forced van der Pol equation (see [2]). In that case one can prove that the only stable oscillations are those which correspond to the upper part of the curve between vertical tangents. In other words, for values of ω outside of the vertical tangents, there is no stable oscillation with frequency ω. Thus, this is where the so-called "locking-in" phenomenon does not occur.

No attempt has been made to prove which oscillations are stable. This would require

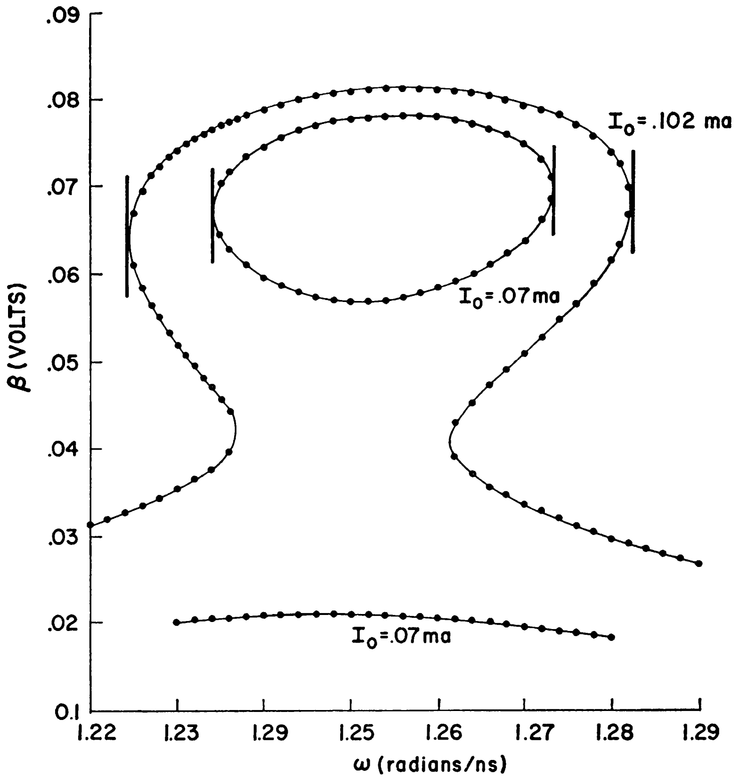
.09
.08
.07
.06
.05
.04
.03
.02
0.1
β (VOLTS)
1.22
1.23
1.29
1.25
1.26
1.27
1.28
1.29
ω (radians/ns)
I_0=.102 ma
I_0=.07ma
I_0=.07ma

FIG. 7.

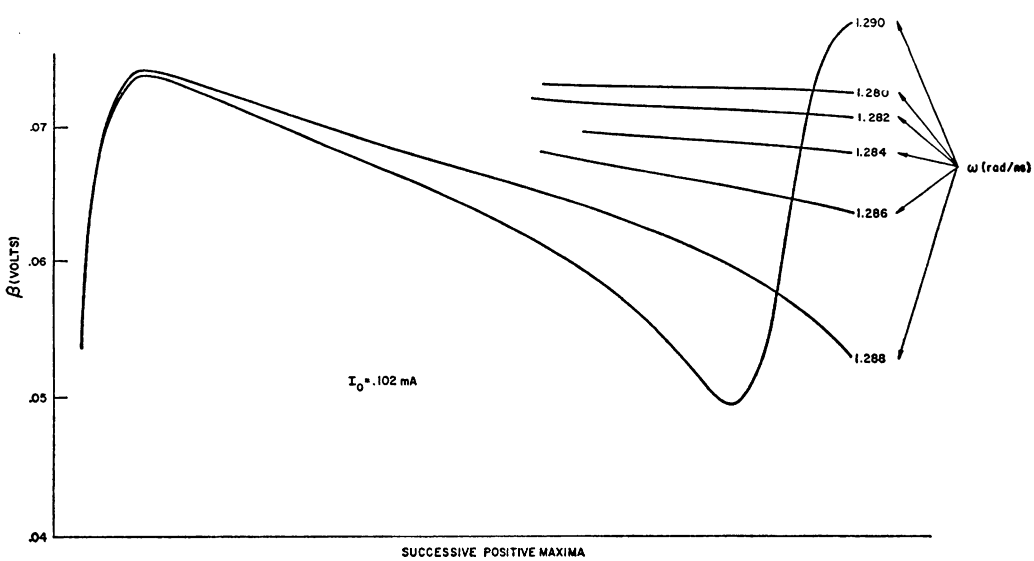
.07
.06
.05
.04
β (VOLTS)
1.290
1.280
1.282
1.284
1.286
1.288
ω (rad/ns)
I_0=.102 mA
SUCCESSIVE POSITIVE MAXIMA

FIG. 8.

a Floquet theory for neutral difference-differential equations with periodic coefficients, and no such theory is available at this time. Such a theory does exist for functional-differential equations which include the retarded but not the neutral case, and this is described in [7] and [8].

As in the autonomous case, some computer solutions were obtained. The results are shown in Fig. 8 where successive maxima are plotted for different values of ω when $I_0 = .102$ m.a. Note that where the curve is relatively flat, the maximum voltages are in good agreement with the values of β in Fig. 7. It is interesting that as ω is increased beyond the vertical tangent, i.e., 1.282 rad./n.s., another frequency appears and seems to modulate the amplitude. This indicates that the frequency-amplitude relation (7.4) does predict approximately the range of "locking-in". Also, Fig. 8 shows how "locking-in" ceases to exist; i.e., another frequency appears and the oscillation is, in general, almost periodic.

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