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ON NONOSCILLATING NETWORKS

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1. In a recent paper on electric networks [1] we investigated the asymptotic behavior of the solutions of nonlinear differential equations of the form

$$\begin{aligned} \dot{x} &= -Ax + By, \\ \dot{y} &= Cx - f(y). \end{aligned} \tag{1}$$

Here x is an n-vector, y an m-vector, A, B, C constant matrices and $f(y)$ a vector function with m components. In particular, we gave criteria which ensure that every solution of the above system approaches an equilibrium of which several may be present. In fact, the case of several equilibrium solutions is of main interest in applications to flip-flop circuits, for which one wants to guarantee that solutions fall into one of the allowed equilibrium points and do not oscillate.

Such criteria can be derived, for example, with the aid of a so-called Liapounov function, which was constructed for systems for which

$$\begin{gathered} A = A^T, \qquad B = -C^T, \\ \left(\frac{\partial f_k}{\partial y_l}\right) \text{ symmetric}. \end{gathered} \tag{2}$$

These conditions reflect the reciprocity of the network. For such systems we derived the following simple criterion: Introduce the scalar function $G(y)$ by

$$\operatorname{grad}_y G = B^T A^{-1} B y + f(y)$$

and assume that

$$G(y) \to \infty \quad \text{as} \quad |y| \to \infty$$

and G possesses only finitely many critical points. Then it can be shown that (see [1, Theorem 3, p. 19]) every solution of (1) approaches one of the critical points of G as $t \to \infty$ provided A is positive definite, and

$$||A^{-1}B|| < 1. \tag{3}$$

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In this paper we shall show that such results can be obtained with the use of transfer functions—avoiding the construction of a Liapounov function. This approach leads to more general criteria and probably to more useful ones. We proceed as follows: Eliminating x from (1) we find for y an integral equation of the form

$$\dot{y} + K\dot{y} = -g(y) + \gamma(t) \tag{4}$$

where

$$K\phi = \int_0^t k(t-\tau)\phi(\tau)\,d\tau; \qquad k(t) = CA^{-1}e^{-At}B, \qquad g(y) = f(y) - CA^{-1}By,$$

$$\gamma(t) = Ce^{-At}a \text{ with a constant vector } a^1.$$

This leads us to the study of system (4) for which the kernel $k(t)$ as well as $\gamma(t)$ decays exponentially as $t \to +\infty$. We shall be interested in comparing the asymptotic behavior of solutions of (4) with those of

$$\dot{z} = -g(z). \tag{5}$$

We shall investigate the integro-differential equation (4) on its own merits and later discuss the special situation which derives from (1) and (2).

It is well known that if $g(z)$ is the gradient of a function $G(z)$ which tends to ∞ as $|z| \to \infty$ and has finitely many critical points then every solution of (5) tends to one of those critical points of G. We shall show that Eq. (4) has the same property provided that the Laplace transform

$$\hat{k}(s) = C(sA + A^2)^{-1}B$$

of $k(t)$ satisfies

$$\operatorname{Re}\{\langle\bar{\eta}, \eta\rangle + \langle\bar{\eta}, \hat{k}(s)\eta\rangle \geq \delta > 0 \quad \text{for} \quad |\eta| = 1 \tag{6}$$

and all purely imaginary s, for some $\delta > 0$.

In the special case of a reciprocal network, i.e. if (2) holds, the above condition (6) reduces to (3) since the harmonic function

$$\operatorname{Re}\langle B\eta, (sA + A^2)^{-1}B\eta\rangle$$

in $\operatorname{Re} s \geq 0$ takes its maximum at $s = 0$. Thus it suffices to check (6) for $s = 0$.

We shall describe other criteria which guarantee asymptotic approach to equilibria. In the special case when only a single equilibrium is approached one speaks of asymptotic stability. Criteria for absolute stability have been studied extensively by Popov (see for example [2]). It was our aim to make use of Popov's ideas (frequency method) for the study of nonoscillatory networks. The applications of Popov's method to our problem is, in fact, quite straightforward. It is based on the simple observation that even if there are several equilibria present the derivatives $\dot{x}$, $\dot{y}$ will approach the single point (0, 0). We also want to point out that we allow a vector function $f(y)$ while the theory of absolute stability is usually restricted to $m = 1$, i.e. scalar functions $f(y)$.[2] The additional integrability condition, that g is the gradient of a single function is a natural consequence of the reciprocity of networks.

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[1] $a = x(0) - A^{-1}By(0)$.

[2] See, however, [4].

2. Asymptotic behavior. We turn to the study of the Eq. (4) where we assume that $g(y)$ is the gradient of a function $G(y)$ which tends to ∞ as $|y| \to \infty$. Without loss of generality we can require $G \geq 0$ for all y.

LEMMA 1. *If $\gamma \in L_2(0, \infty)$, $k(t) \in L_2(0, \infty)$ and the Laplace transform $\hat{k}(s)$ of $k(t)$ satisfies* (6), *then any solution y of* (4) *exists for all $t \geq 0$ and one has*

$$\int_0^\infty \dot{y}^2 \, dt \leq c; \qquad |y(t)| \leq c,$$

where c depends on the initial value of $y(0)$.

Proof. This results immediately from the fact that

$$\int_0^T \langle g(y), \dot{y} \rangle \, dt = G(y) |_0^T \geq -G(y(0)).$$

Hence multiplying (4) by $\dot{y}$ and integrating yields

$$\int_0^T \langle (\dot{y} + K\dot{y}), \dot{y} \rangle \, dt + G(y) |_T = \int_0^T \langle \gamma(t), \dot{y} \rangle \, dt + G(y(0)).$$

On the other hand, the inequality (6) together with Parseval's equation yields

$$\int_0^T \langle (\dot{y} + K\dot{y}), \dot{y} \rangle \, dt \geq \delta \int_0^T \langle \dot{y}, \dot{y} \rangle \, dt$$

and thus

$$\delta \int_0^T |\dot{y}|^2 \, dt + G(y) |_T \leq \int_0^T \langle \gamma, \dot{y} \rangle \, dt + G(y(0)).$$

Estimating the integral on the right in a standard fashion by

$$\int_0^T \langle \gamma, \dot{y} \rangle \, dt \leq \frac{\delta}{2} \int_0^T |\dot{y}|^2 \, dt + \frac{1}{2\delta} \int_0^T |\gamma|^2 \, dt$$

yields

$$\frac{\delta}{2} \int_0^T |\dot{y}|^2 \, dt + G(y(T)) \leq \frac{1}{2\delta} \int_0^T |\gamma|^2 \, dt + G(y(0)).$$

Since $G \geq 0$ it follows that $\dot{y} \in L_2(0, \infty)$. Moreover, since $G \to \infty$ as $|y| \to \infty$ we conclude that y is bounded, which proves the lemma.

Of course, from Lemma 1 we cannot conclude directly that $\dot{y}$ tends pointwise to zero. However, it follows readily that y approaches pointwise an equilibrium point of G. We shall show more generally:

LEMMA 2. *Let y be a solution of* (4) *where we assume that*

$$\gamma(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \int_0^\infty |k(t)|^2 \, dt < \infty$$

and that

$$\int_0^\infty |\dot{y}|^2 \, dt \leq c; \qquad \sup |y| \leq c.$$

Then the limit set of y agrees with that of a solution z of

$$\dot{z} = -g(z).$$

Proof. The translates $y(t + \tau)$ form an equicontinuous family of functions since—by the Schwarz inequality—

$$|y(t'') - y(t')| \leq \int_{t'}^{t''} |\dot{y}| \, dt \leq c \, |t'' - t'|^{1/2}.$$

Therefore, there exists a sequence $\tau_n \to \infty$ such that $y(t + \tau_n)$ converges uniformly to a continuous vector function $z(t)$. This function satisfies the differential equation

$$\dot{z} = -g(z)$$

as we shall show readily.

For this purpose we show that

$$\int_0^t k(t - \tau)\dot{y}(\tau) \, d\tau \to 0 \quad \text{as} \quad t \to \infty. \tag{7}$$

Indeed, given $\epsilon > 0$ there exists a $T = T(\epsilon)$ such that

$$\int_T^\infty |\dot{y}|^2 \, dt < \epsilon \left(\int_0^\infty |k|^2 \, dt \right)^{-1}.$$

Hence, for $t > T$

$$\left| \int_0^t k(t - \tau)\dot{y}(\tau) \, d\tau \right| \leq \int_0^T |k(t - \tau)\dot{y}(\tau)| \, d\tau + \left(\int_T^t |k(t - \tau)|^2 \, d\tau \int_T^t |\dot{y}|^2 \, d\tau \right)^{1/2}$$

$$\leq \left(\int_{t-T}^\infty |k(t)|^2 \, dt \right)^{1/2} c^{1/2} + \epsilon.$$

Fixing T we can choose t so large that the first term becomes less than ϵ.

Since also $\gamma(t) \to 0$ as $t \to \infty$ we see that formally Eq. (4) goes into (5). However, since z is not known to be differentiable we have to argue more carefully: Let Δ denote the interval $(t, t+h)$ with some $h > 0$. Integrating (4) over the interval $(t+\tau_n, t+\tau_n+h)$ we find

$$\int_\Delta (\dot{y}_n + K\dot{y}|_{t'+\tau_n}) \, dt' = -\int_\Delta g(y_n) \, dt' + \int_\Delta \gamma(t' + \tau_n) \, dt'$$

where $y_n(t') = y(t' + \tau_n)$. Since $y_n \to z$ uniformly we conclude from (7) that

$$\frac{z(t + h) - z(t)}{h} = -\frac{1}{h} \int_\Delta g(z(t')) \, dt'$$

hence with $h \to 0$

$$\dot{z} = -g(z).$$

If we denote the limit set of y by Y and that of z by Z, then it is clear that $Y \supset Z$. Indeed, if $\zeta \in Z$ there exists a sequence $t_k \to \infty$ such that $z(t_k) \to \zeta$. Since $z(t_k) = \lim_{n\to\infty} y(\tau_n + t_k)$ one can determine a sequence n_k such that

$$y(\tau_{n_k} + t_k) \to \zeta$$

so that $\zeta \in Y$.

But also, conversely, every $\eta \in Y$ belongs to Z as we show now: There exists a sequence $t_k \to \infty$ such that $y(t_k) \to \eta$. Now let $z(t)$ be that solution of (5) which satisfies

$z(0) = \eta$. Then we know that $y(t_k + t) \to z(t)$ as $k \to \infty$ for $t \geq 0$. If we choose a subsequence t_{k_n} of the t_k such that

$$t_{k_n} > 2t_n$$

then it is clear that with $t = t_{k_n} - t_n$

$$|y(t_n + t) - z(t)| = |y(t_{k_n}) - z(t_{k_n} - t_n)| \to 0$$

as $n \to \infty$. Since $t_{k_n} - t_n > t_n \to \infty$ it follows that

$$\eta = \lim_{n\to\infty} z(t_{k_n} - t_n),$$

i.e., $\eta \in Z$, which proves $Y = Z$.

In particular, if g is a gradient of a function with a finite number of critical points it is well known that the limit set of a solution $z(t)$ of (5) is precisely one critical point. Therefore we conclude:

THEOREM 1. *If in* (4) *the vector function g is the gradient of a function G with finitely many critical points, $G(y) \to \infty$ as $|y| \to \infty$ and if $\gamma, k \in L_2$, $\gamma \to 0$ as $t \to \infty$, then any solution $y(t)$ approaches a critical point of G provided that*

$$\text{Re}\ \{\langle\bar{\eta}, \eta\rangle + \langle\bar{\eta}, \hat{k}(s)\eta\rangle\} \geq \delta\ |\eta|^2 \quad \textit{with} \quad \delta > 0.$$

From here it is immediately shown that the solutions of (1) approach equilibrium solutions provided the eigenvalues of A lie in the right half plane. Indeed

$$x = e^{-At}x_0 + \int_0^t e^{-A(t-\tau)}By(\tau)\, d\tau$$

and the convergence of y for $t \to \infty$ implies that of $x(t)$.

3. Another criterion. In some cases it is possible to give a lower bound for the form

$$\sum_{k,l=1}^{m} \frac{\partial f_k}{\partial y_l} \eta_k \eta_l \geq -\lambda \sum_{k=1}^{m} \eta_k^2, \tag{8}$$

and $-\lambda$ measures the amount of "negative resistance."

If such a bound is available then one can replace condition (6) by a milder one, which we will derive now.

For this purpose we differentiate the relation (4) and get

$$\ddot{y} + \dot{K}\dot{y} + f_y(y)\dot{y} = \dot{\gamma}, \tag{9}$$

where f_y is the matrix formed by the elements $\partial f_k/\partial y_l$ and

$$\dot{K}\phi = \int_0^t \dot{k}(t - \tau)\phi(\tau)\, d\tau.$$

We used here

$$\frac{d}{dt}(K\phi) = \dot{K}\phi + k(0)\phi(t).$$

Now we proceed with Eq. (9) similarly as before with Eq. (4): Multiplying both sides with $\dot{y}$ and integrating we find with (8)

$$\tfrac{1}{2}\ |\dot{y}|^2|_0^T + \int_0^T \langle\dot{y}, \dot{K}\dot{y}\rangle\, dt - \lambda \int_0^T |\dot{y}|^2\, dt \leq \int_0^T \langle\dot{\gamma}, \dot{y}\rangle\, dt$$

or, after a short calculation,

$$\int_0^T \langle \dot{y}, (\dot{K} - \lambda' I)\dot{y}\rangle \, dt \le \tfrac{1}{2} \, |y(0)|^2 + \frac{1}{4\delta} \int_0^T |\dot{\gamma}|^2 \, dt$$

with $\lambda' = \lambda + \delta$, $\delta > 0$. We multiply this inequality with $\rho \ge 0$ and combine it with the previously derived one

$$\int_0^T \langle \dot{y}, (I + K)\dot{y}\rangle \, dt + G(y(T)) \le \int_0^T \langle \gamma, \dot{y}\rangle \, dt + G(y(0))$$

to get

$$\int_0^T \langle \dot{y}, L\dot{y}\rangle \, dt + G(y(T)) \le \frac{1}{4\delta}\left\{\int |\gamma|^2 \, dt + \rho \int |\dot{\gamma}|^2 \, dt\right\} + G(y(0)) + \frac{\rho}{2} (y(0))^2 = C_1 \quad (10)$$

with

$$L = ((1 - \delta)I + K) + \rho(\dot{K} - \lambda' I).$$

If we now require that

$$\int_0^\infty \langle \phi, L\phi\rangle \, dt \ge \delta \int_0^\infty |\phi|^2 \, dt \quad \text{for all} \quad \phi \in L_2(0, \infty) \quad (11)$$

we can conclude again that $y \in L_2(0, \infty)$. The rest of the proof of the asymptotic approach is just as before.

The above condition can be easily expressed in terms of the Laplace transform $\hat{k}(s)$ of k. Indeed, (11) will hold if

$$\mathrm{Re} \, \langle \bar{\eta}, (I + \hat{k}(s) + \rho[s\hat{k}(s) - k(0) - \lambda' I])\eta\rangle \ge 2\delta \, |\eta|^2, \quad \text{for} \quad \mathrm{Re}\, s = 0,$$

i.e. the Hermitian part of

$$I + \hat{k}(s) + \rho[\hat{k}(s) - k(0) - \lambda' I]$$

is positive. Introducing the function

$$H(s) = s\hat{k}(s) - k(0) \quad (12)$$

this condition takes the form

$$\mathrm{Re} \left\langle \bar{\eta}, \left[(1 - \rho\lambda')I + \left(\rho + \frac{1}{s}\right)H(s) + \frac{k(0)}{s}\right]\eta \right\rangle > 2\delta \, |\eta|^2, \qquad \mathrm{Re}\, s = 0. \quad (13)$$

Following the same lines as in Sec. 2 we obtain

THEOREM 2. *Assume in* (4) *that g is the gradient of a function G with the properties mentioned before and that $\dot{\gamma}, k \in L_2$ and that $\gamma, \dot{\gamma} \to 0$ for $t \to +\infty$. Then any solution of* (4) *approaches an equilibrium as $t \to +\infty$ provided* (13) *holds for some $\rho \ge 0$, $\delta > 0$ and all purely imaginary s.*

4. Discussion of the condition (13). We want to give condition (13) a more geometrical form which makes it evident that it is weaker than condition (6). We will assume that

$$k(0) = k^*(0)$$

so that $k(0)$ drops out of (13) since s is purely imaginary. We compute that in our case

$$H(s) = -C(sI + A)^{-1}B$$

is a rational matrix function which for $s \to \infty$ tends to zero. We can give a geometric discussion similar to that of Popov (see [3]) if we introduce the real functions

$$\operatorname{Re} \langle \bar{\eta}, H(s)\eta \rangle = \phi(s),$$

$$\frac{\operatorname{Im} \langle \bar{\eta}, H(s)\eta \rangle}{\operatorname{Im} s} = \psi(s)$$

for a fixed complex vector η with $|\eta| = 1$, so that

$$\langle \bar{\eta}, H(s)\eta \rangle = \phi + s\psi \quad \text{for} \quad \operatorname{Re} s = 0.$$

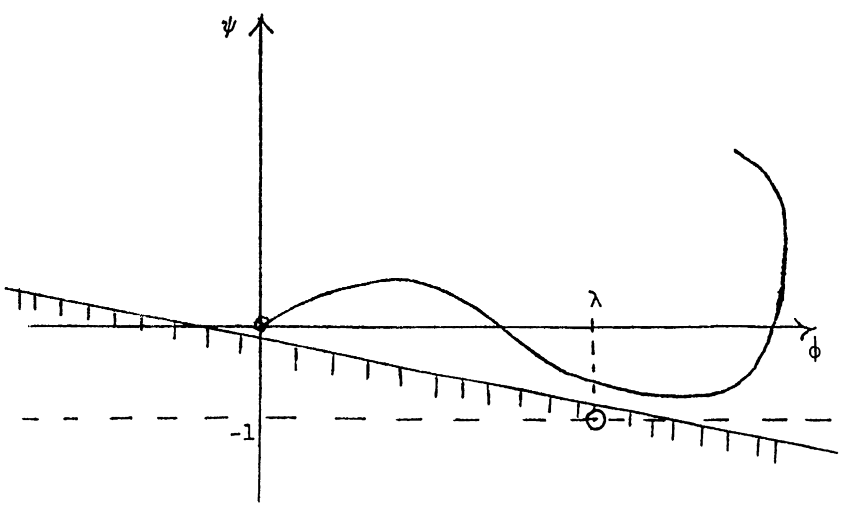
The inequality (13) amounts to

$$1 - \rho\lambda' + \rho\phi + \psi \geq 2\delta$$

or

$$(\psi + 1) + \rho(\phi - \lambda') \geq 2\delta \tag{14}$$

for all purely imaginary s, and all complex η with $|\eta| = 1$. This means geometrically that in the (ϕ, ψ)-plane there should exist a line passing above the point $(\phi, \psi) = (\lambda', -1)$ at a negative slope below the graph of $(\phi(s), \psi(s))$.



Since (0, 0) lies on the graph we have to choose a line which passes below the origin. Dropping the irrelevant small $\delta > 0$ we can express (13) as follows: There should exist some straight line passing above the point $P = (\lambda, -1)$ and below the origin (0, 0) and such that the graph $(\phi(s), \psi(s))$ stays above the curve. It is clear that for $\lambda \to \infty$ the slope ρ has to approach 0 and we come back to the criterion of Sec. 2.

We turn to the reciprocal case, i.e. to system (1) satisfying (2). In this case we have

$$H(s) = B^T(sI + A)^{-1}B$$

and with

$$\zeta = (sI + A)^{-1}B\eta$$

we find

$$\phi = \langle \bar{\zeta}, A\zeta \rangle; \qquad \psi = -|\zeta|^2. \tag{15}$$

A sufficient condition for (14) to hold is that either

$$\begin{array}{lll} & \text{(a)} & (A - \lambda I) \quad \text{is positive definite,} \\ \text{or} & & \\ & \text{(b)} & ||A^{-1}B|| < 1. \end{array} \tag{16}$$

Namely, in the first case one can choose ρ so that

$$\lambda < \frac{1}{\rho} < \alpha_{\min}$$

where $\alpha_{\min}$ is the smallest eigenvalue of A. Then

$$\rho A - I \text{ is positive definite}$$

and

$$\begin{aligned}(\psi + 1) + \rho(\phi - \lambda) &= -|\zeta|^2 + 1 + \rho\langle \bar{\zeta}, A\zeta \rangle - \rho\lambda \\ &= \langle \bar{\zeta}, (\rho A - I)\zeta \rangle + 1 - \rho\lambda \geq 1 - \rho\lambda > 0\end{aligned}$$

since also $\rho\lambda < 1$. On the other hand, if $||A^{-1}B|| < 1$, then we choose $\rho = 0$ and because of the symmetry of A we have

$$|\zeta| = |(sI + A)^{-1}B\eta| \leq |A^{-1}B\eta| \quad \text{for} \quad \operatorname{Re} s = 0.$$

Indeed,

$$|\langle(\bar{s}A^{-1} + I)\bar{\zeta}, \zeta\rangle| \geq |\zeta|^2$$

hence by the Schwarz' inequality

$$|\zeta| \leq |(sA^{-1} + I)\zeta| = |A^{-1}B\eta|.$$

Therefore by assumption $|\zeta| < |\eta| = 1$, i.e.

$$\psi + 1 = -|\zeta|^2 + 1 > 0$$

which verifies (14) for $\rho = 0$.

Incidentally, if λ exceeds the largest eigenvalue of A then our condition (14) implies $||A^{-1}B|| < 1$. This follows from

$$\begin{aligned}0 < \psi + 1 + \rho(\phi - \lambda) &= -|\zeta|^2 + 1 + \rho\langle \bar{\zeta}, A\zeta \rangle - \rho\lambda \\ &\leq -|\zeta|^2 + 1 + \rho\lambda\, |\zeta|^2 - \rho\lambda \\ &= (1 - \rho\lambda)(1 - |\zeta|^2).\end{aligned}$$

This relation has to hold for all purely imaginary s. For $s \to \infty$ we have $\zeta \to 0$, hence $1 - \rho\lambda > 0$ and therefore

$$|\zeta| < 1$$

or

$$|A^{-1}B\eta| < 1 \quad \text{for all} \quad |\eta| = 1.$$

Thus the condition (14) may be sharper than (16) only if λ lies between the smallest and largest eigenvalue of A.

The conditions (16) agree with those derived in [1] for reciprocal networks, while the present conditions (14) have a somewhat wider applicability, as we do not have to require A to be symmetric and $B = -C^T$.

Of course, the requirement that g is a gradient is very stringent (if $m > 1$) since it implies the symmetry of the matrix

$$f_y - CA^{-1}B$$

for all y. This shows on the other hand that the reciprocal networks are particularly suited for this treatment.

It remains undecided whether under more general conditions also a Liapounov function can be constructed, although this seems quite likely.

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