A NUMERICAL DETERMINATION OF SUBHARMONIC RESPONSE FOR THE DUFFING EQUATION $\ddot{x} + \alpha x + \beta x^3 = F \cos \omega t$ $(\alpha > 0)$*

BY

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1. Introduction. It is known [1], [3], [4] that the Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = F \cos \omega t \qquad (\alpha > 0) \tag{1}$$

under the initial conditions

$$x(0) = A; \qquad \dot{x}(0) = 0 \tag{2}$$

possesses periodic solutions of period $2n\pi/\omega$ or circular frequency ω/n for all positive integral n and that these solutions together with their corresponding frequencies are analytic in β for arbitrary A, F and $\alpha > 0$ provided that β is sufficiently small. In the author's doctoral dissertation [3] and its abridgment, a classification of the periodic solutions was made in which it was shown that there are *several* types of periodic solutions, two types of period $2\pi/\omega$, the ordinary harmonic and ultraharmonic, and two types of period $2n\pi/\omega$ $(n = 2, 3, \cdots)$, the ordinary subharmonic and ultra-subharmonic. More precisely, after putting the differential equation in nondimensional form

$$\nu^2 \xi'' + \xi + \xi^3 = \cos \theta \tag{3}$$

under the initial conditions

$$\xi(0) = M; \qquad \xi'(0) = 0 \tag{4}$$

where the differentiation refers to the variable $\theta = \omega t$, and where

$$\nu^2 = \omega^2/\alpha, \qquad \delta = \beta F^2/\alpha^3, \qquad M = A\alpha/F, \qquad \xi = \alpha x/F, \tag{5}$$

we have the following classification for the periodic solutions of Eq. (3) under the initial conditions (4).

The ordinary harmonic solutions are those solutions of period 2π for which ν^2 reduces to $1 - 1/M$ when $\delta = 0$.

The ultraharmonic solutions are those solutions of period 2π for which ν reduces to $1/m$ $(m = 2, 3, \cdots)$ when $\delta = 0$.

The ordinary subharmonic solutions are those solutions of smallest period $2n\pi$ $(n = 2, 3, \cdots)$ for which ν reduces to n when $\delta = 0$.

The ultra-subharmonic solutions are those solutions of smallest period $2n\pi$ $(n = 2, 3, \cdots)$ for which ν reduces to n/m where n and m are relatively prime and $\neq 1$ when $\delta = 0$.

The following approximate response relation, the relation among the parameters yielding periodic solutions, was obtained for the ultraharmonics, ordinary subharmonics and ultra-subharmonics:

$$\nu^2 = (p/q)^2 \left\{ 1 + \tfrac{3}{4}\delta \left[M^2 + \frac{2M}{(p/q)^2 - 1} + \frac{3}{\{(p/q)^2 - 1\}^2} \right] \right\} (p/q \neq 3, \tfrac{1}{3}, 1). \tag{6}$$

Since the approximate relation (6) is readily obtained by a simple application of the perturbation procedure [3], [4], we shall refer to the corresponding curves as the perturbation curves.

It is the purpose of this note to present the results of the author's numerical digital computer determination of the ordinary subharmonic solutions with smallest period 4π, for which $p/q = 2$ and their response curves for values of δ in the interval $0 < \delta \leq 10$ and values of M in the interval $-10 \leq M \leq 10$ and values of ν suitably determined to insure the periodicity of the solutions, and finally to compare these curves with the corresponding perturbation response curves. We shall refer to the curves determined by the computer as the computer curves. These results will be considered in Sec. 3.

2. The numerical method. The differential Eq. (3) is put in the form

$$\frac{d\xi}{d\theta} = \eta; \qquad \frac{d\eta}{d\theta} = \frac{-\xi - \delta\xi^3 + \cos\theta}{\nu^2} \qquad (7)$$

under the initial conditions

$$\xi(0) = M; \qquad \xi'(0) = 0. \qquad (8)$$

The interval $0 \leq \theta \leq 4\pi$ is subdivided into 400 equal subintervals by the points $\theta_n = .01n\pi$ $(n = 0, 1, 2, \cdots 400)$. The values of ξ at θ_i, $i = 1, 2, 3, 4, 5$ and the values of η at θ_j, $j = 1, 2, 3, 4$ are computed from the Taylor series for $\xi(\theta)$ expanded about $\theta = 0$ and its differentiated series respectively. The Taylor expansion was carried out to the
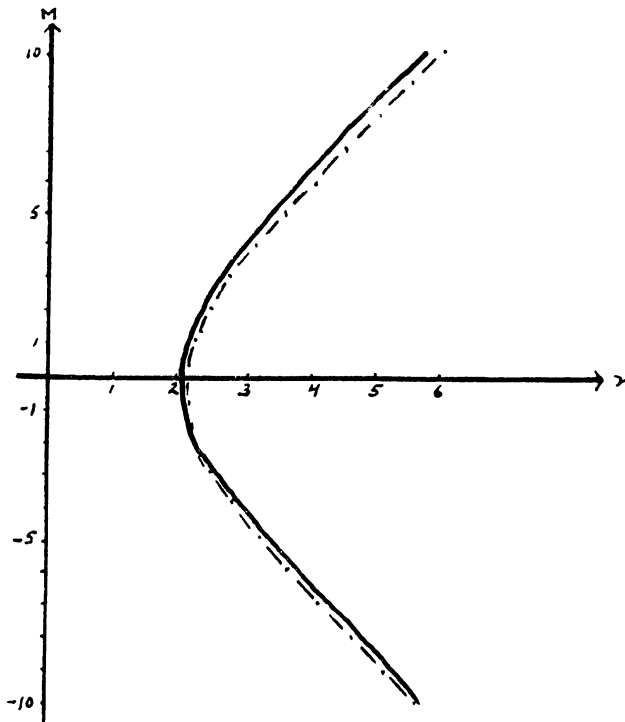

FIG. 1. Comparison between the perturbation and computer response curves for $\delta = .1$. ———— computer curve for periodic solutions of period 4π. ·—·— perturbation curve for the ordinary subharmonic solutions of period 4π for $p/q = 2$.
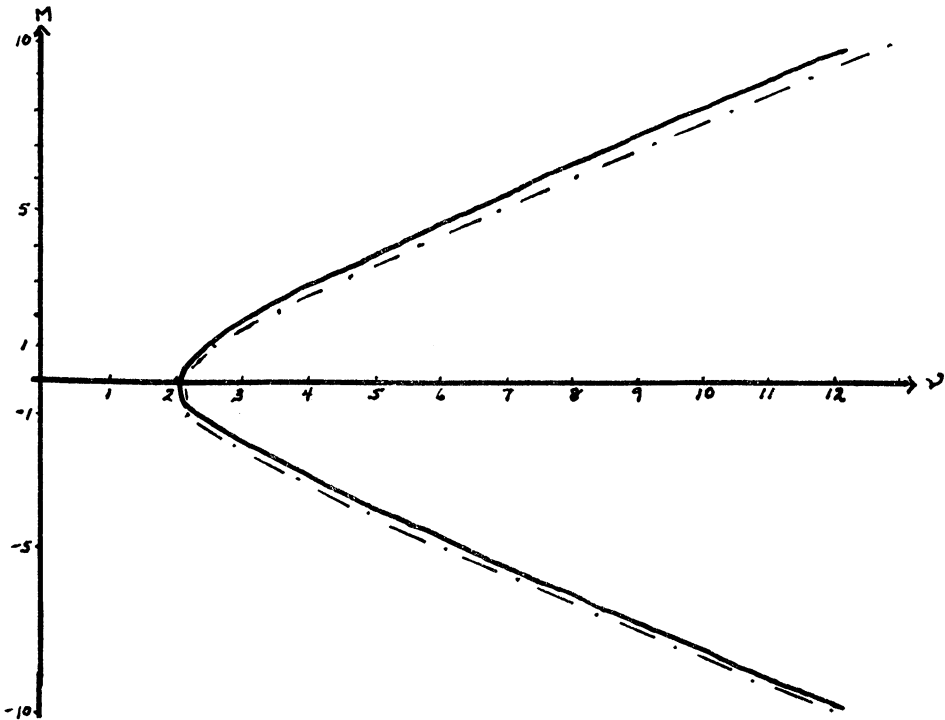
Fig. 2. Comparison between the perturbation and computer response curves for $\delta = .5$. ———— computer curve for periodic solutions of period 4π. $\cdot$———$\cdot$— perturbation curve for the ordinary subharmonic solutions of period 4π for $p/q = 2$.

8th degree term in θ. With the values of ξ and η now known for θ_i, $i = 0, 1, 2, 3, 4, 5$ and θ_j, $j = 0, 1, 2, 3, 4$ respectively, we apply the Adams method [2] to find the values of ξ and η at successive points of the interval using the formulas

$$\xi_{n+1} = \xi_n + h[p_n + \tfrac{1}{2}\Delta p_{n-1} + \tfrac{5}{12}\Delta^2 p_{n-2} + \tfrac{3}{8}\Delta^3 p_{n-3} + \tfrac{251}{720}\Delta^4 p_{n-4} + \tfrac{95}{288}\Delta^5 p_{n-5}] \qquad (9)$$

$$\eta_n = \eta_{n-1} + h[q_{n-1} + \tfrac{1}{2}\Delta q_{n-2} + \tfrac{5}{12}\Delta^2 q_{n-3} + \tfrac{3}{8}\Delta^3 q_{n-4} + \tfrac{251}{720}\Delta^4 q_{n-5}] \qquad (10)$$

where $h = .01\pi$, $p_n = \eta_n = d\xi/d\theta$ at $\theta = \theta_n$, $q_n = d\eta/d\theta$ at $\theta = \theta_n$, $\Delta p_r = p_{r+1} - p_r$, $\Delta^k p_r = \Delta^{k-1} p_{r+1} - \Delta^{k-1} p_r$, with similar expressions for Δq_r and $\Delta^k q_r$. The values of q_i at θ_i, $i = 0, 1, 2, 3, 4$ are computed from the second equation in (7) and then $\eta_5 = p_5$ is calculated from equation (10). The value of p_5 is then inserted in equation (9) and ξ_6 is calculated. This process is repeated until all the values of ξ_i and η_i are computed.

The procedure now is to assign fixed values for M, δ and ν, compute the solution and then, keeping the same fixed values for M and δ, to vary the ν until $\xi'(2\pi) = 0$. The last equality is the condition [3] that the solution have the period 4π. Since $\xi'(2\pi)$, for fixed δ and M, is a continuous function of ν, we followed the programmer's suggestion and used the method of false position [2] which is easily adapted to the computer, to find the solution of the equation $\xi'(2\pi) = f(\nu) = 0$ to a specified degree of accuracy. The computer was programmed, using 8-place mantissas, to find ν such that $|\xi'(2\pi)| < 10^{-5}$. When the value of ν satisfying this condition was found, the computer was instructed to print out the solution after performing the calculations using 28-place mantissas. As a result, in many cases the values of $|\xi'(2\pi)|$ were greater than 10^{-5} but always less
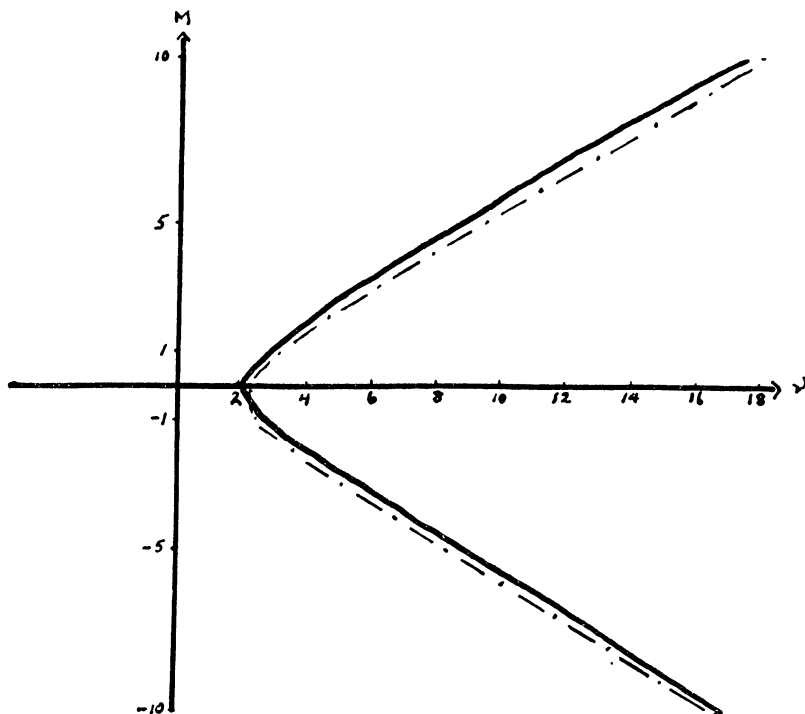
FIG. 3. Comparison between the perturbation and computer response curves for $\delta = 1$. ———— computer curve for periodic solutions of period 4π. ·—·— perturbation curve for the ordinary subharmonic solutions of period 4π for $p/q = 2$.

than 10^{-4}. The solution was then checked to be s. re that $\xi'(\pi)$ was not too close to zero, since $\xi'(\pi) = 0$ would mean that the solution ha.l its period equal to 2π. A check on the reliability of the computer was made by examining the values of ξ and ξ' at 4π, the end point of the interval, where their values should be the same as their respective initial values. There was excellent agreement in all cases; for example, for $\delta = 10$, all the values of $\xi(4\pi)$ were *exactly* the same as their initial values except in the case of $M = 0$, where the value of $\xi(4\pi)$ was equal to 4.5294370×10^{-6} and all values of $|\xi'(4\pi)|$ were less than 10^{-4}, five were less than 10^{-5} and one was less than 10^{-6}.

3. Results. The results obtained are shown in Figs. 1 through 6. Figs. 1 through 5 show a comparison between the computer curves and perturbation curves for $p/q = 2$ in the νM-plane for different values of δ. Fig. 6 shows the computer curves in the $\nu\delta$-plane for different values of M. An inspection of Figs. 1 through 5 shows excellent agreement between the two sets of curves. More precisely, using the relative error $|\nu_c - \nu_p|/\nu_c$ as a measure for comparison, where ν_c and ν_p are the values of ν obtained from the computer and the perturbation formula (6) respectively, we found the following results: The agreement is best for $\delta = .1$ (Fig. 1), where the largest relative error is less than 5 percent occurring for $M = 8$. The agreement is very good for all values of δ (Figs. 1 through 5) when M is negative, the largest relative error being approximately 13 percent and occurring for $\delta = 10$ and $M = -1$. For the larger values of δ (Figs. 3, 4, 5), the agreement is best for the larger negative values of M, the relative error being approximately 1 percent for $M = -10$. The agreement is not very good for $\delta = 10$ and $M = 0$, where the largest relative error of approximately 44 percent occurs.

The agreement for the larger values of δ had not been anticipated. It must be pointed out that the existence of our periodic solutions had been established [3] only for sufficiently small δ. It is therefore both interesting and surprising that the approximate perturbation relation (6), in which the terms involving powers of δ higher than the first have been neglected, yields results that are very good even when δ is not small.

Another significant result is the following: The perturbation procedure assumes a power series development in δ for ν

$$\nu = \nu_0 + \nu_1 \delta + \cdots \tag{11}$$

from which

$$\nu^2 = \nu_0^2 + 2\nu_0\nu_1 \delta + \cdots . \tag{12}$$

The perturbation values of ν were computed from Eq. (6) by taking the square root of the right side of (6), that is, ν was calculated as $(\nu_0^2 + 2\nu_0\nu_1 \delta)^{1/2}$ not as $\nu_0 + \nu_1 \delta$. In fact, if ν had been calculated as $\nu_0 + \nu_1 \delta$, there would have been no agreement at all between the two sets of curves for the larger values of δ or the larger values of M. It thus appears that it is ν^2 and not ν which is the essential parameter in the perturbation method applied to the Duffing equation (3) under the initial conditions (4).

One of the interesting questions concerning the various types of periodic solutions of the Duffing equation, which we hope to investigate, is the question of bifurcation, that is, do the various types of periodic solutions merge with each other for suitable values
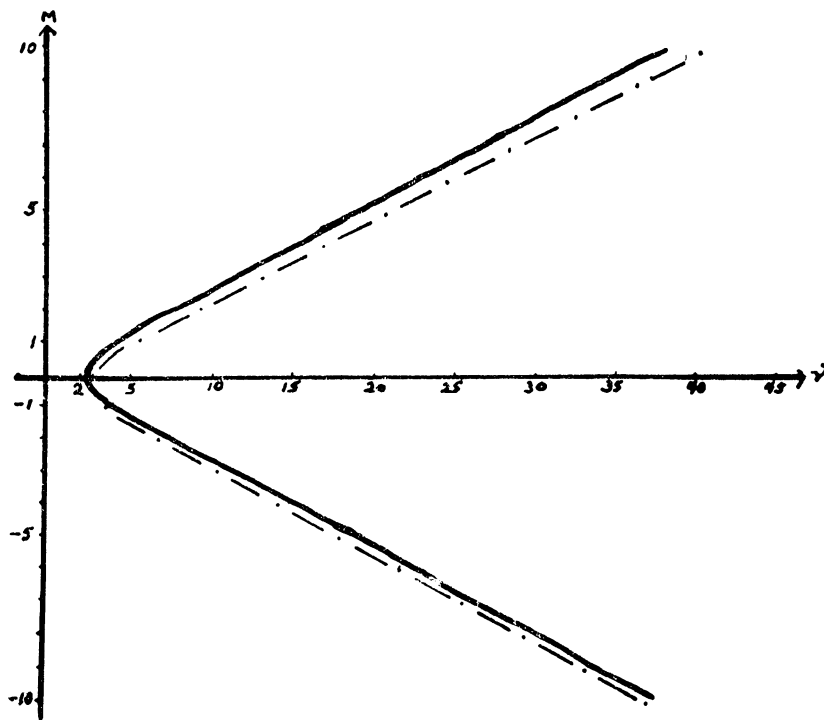

FIG. 4. Comparison between the perturbation and computer response curves for $\delta = 5$. ———— computer curve for periodic solutions of period 4π. ·—·— perturbation curve for the ordinary subharmonic solutions of period 4π for $p/q = 2$.

of the parameters? or in terms of the response curves in the νM-plane for fixed δ, do the response curves for the different types of periodic solutions intersect each other? We have shown [3] that for sufficiently small δ the ordinary harmonic solutions bifurcate with all the other types of periodic solutions. It seems reasonable, in the light of the results that we have obtained for larger values of δ in our present investigation, that bifurcation also occurs for larger values of δ. Is it also possible that the ultraharmonics,
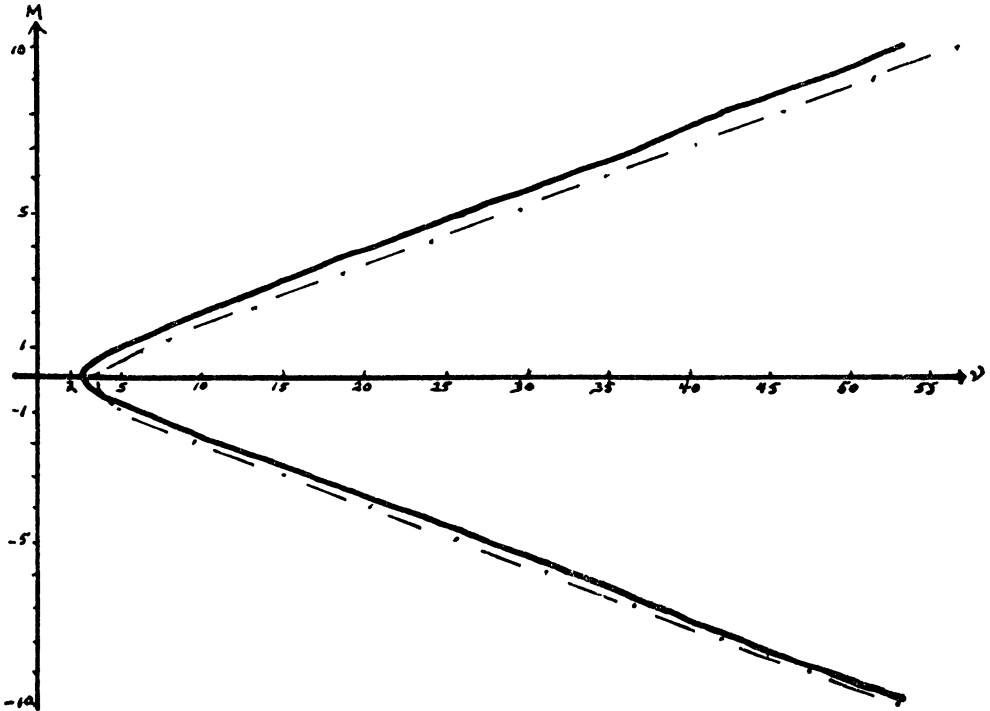

Fig. 5. Comparison between the perturbation and computer response curves for $\delta = 10$. ———— computer curve for periodic solutions of period 4π. ·—·— perturbation curve for the ordinary subharmonic solutions of period 4π for $p/q = 2$.
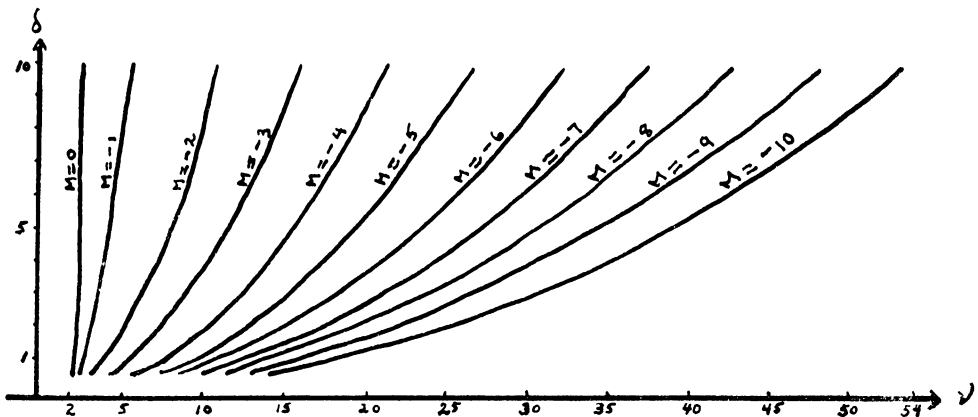

Fig. 6. Computer response curves for periodic solutions of period 4π of the ordinary subharmonic type.

ordinary subharmonics and ultra-subharmonics bifurcate with each other? I wish to thank Professor K. O. Friedrichs of the Courant Institute of Mathematical Sciences for suggesting this possibility. I also wish to thank my colleagues, Professor L. Heil, who made the computer available to me at Brooklyn College, and Mr. H. Givner who did the programming.

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