PHASE SHIFT AND LOCKING-IN REGIONS*

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1. Introduction. The phenomenon of locking-in or entrainement occurs when a system having a natural oscillation of a particular frequency ω_n is forced (with small forcing) at a slightly different frequency. The response resembles the natural oscillation qualitatively, but the frequency of the response is that of the forcing. The natural response is said to "lock in" with the forcing frequency.

There is but a limited range of frequencies about the natural frequency within which locking-in will occur, and the size of this frequency range grows with amplitude of forcing. There is often the theoretical possibility of periodic response when the ratio of forcing frequency to natural frequency is any rational number p/q, and there is as a result the possibility of locking-in for forcing frequency near to any rational multiple of forcing frequency. However, unless the rational number p/q has both p and q small integers, the phenomenon is difficult to observe.

If the forcing has amplitude ϵ and frequency ω, there are regions in the ω-ϵ plane, emanating from points $\epsilon = 0$, $\omega = (p/q)\omega_n$, within which locking-in occurs. Such regions can be established only for small ϵ, with certain additional requirements on the form of the differential equation describing the system. Within the region emanating from $\epsilon = 0$, $\omega = (p/q)\omega_n$ the locked-in solution has frequency ω/p (which is approximately ω_n/q), and this is a subharmonic of order p with respect to the forcing frequency, and a period resembles q periods of the natural oscillation.

There are numerous applications and manifestations of locking-in in electronics. An intriguing possibility of a natural manifestation of locking-in lies in the diurnal rhythm of physiological behavior. Some experiments suggest that a bodily function, e.g. sleeping, may have a natural period somewhat different from twenty-four hours. The actual twenty-four period is the result of the locking-in of the periodic bodily function with the "forcing frequency" provided by the twenty-four hour period of the rotation of the Earth.

As forcing frequency varies within a particular locking-in region, there will be a change of relative "phase" of forcing and response. This assumes, of course, that some notion of phase can be defined when the oscillations are not sinusoidal. This same phase-shift is noted with linear second-order systems where response is essentially in phase for low forcing frequencies but becomes 180° out of phase for very high forcing frequencies.

It is the particular purpose of this note to study the phase relationships as forcing frequency varies through a locking-in region. The relation is not simple in general, but some fairly definite statements can be made in certain important special cases.

2. Notations. Let the unforced system be described by the real vector system

$$x' = g(x), \qquad (' = d/dt) \tag{2.1}$$

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where $g(x) \in C^2$, and let the natural oscillation be a periodic solution $x_0(t)$ of (2.1). Let $x_0(t)$ have least period 2π, and let the associated variational system

$$y' = g'(x_0(t))y \tag{2.2}$$

have $(n - 1)$ characteristic multipliers of absolute value less than 1 (the nth is 1, corresponding to the periodic solution $x_0'(t)$ of (2.2)). The solution $x_0(t)$ is then isolated and asymptotically orbitally stable. Note that this assumption implies that $g(x)$ is nonlinear.

Every vector solution of

$$z' = -g'(x_0(t))^*z, \tag{2.3}$$

the system adjoint to (2.2), is such that

$$z(t)^*x_0'(t) \equiv \text{const}.$$

Let $z_0(t)$ be the unique 2π-periodic solution of (2.3) such that

$$z_0^*(t)x_0'(t) \equiv 1.$$

Let $f(\theta, x, \epsilon)$ be in C^1 near the orbit of $x_0(t)$, and let it have period 2π in θ. We shall consider periodic solutions of

$$x' = g(x) + \epsilon f(\omega(t - \tau), x, \epsilon) \tag{2.4}$$

for small ϵ and ω near a given rational number p/q.

We shall require one result about periodic solutions of perturbed autonomous systems, which is an adaptation to the present notation of a result in [1].

THEOREM A. *The system*

$$x' = g(x) + \epsilon f(p(t - \tau)/q, x, \epsilon), \tag{2.5}$$

where p/q is a rational number in lowest terms, has a periodic solution of least period $2\pi q$ of the form

$$x = x_0(t) + \epsilon x_1(t) + o(\epsilon) \tag{2.6}$$

if τ is a simple zero of

$$F_1(\tau) = \int_0^{2\pi q} z_0^*(s) f\left(\frac{p}{q}(s - \tau), x_0(s), 0\right) ds. \tag{2.7}$$

The solution is asymptotically stable for small ϵ if and only if $\epsilon F_1'(\tau) < 0$. In (2.6) $x_1(t)$ is a uniquely defined function of period $2\pi q$.

REMARK. The periodic solution found in Theorem A has its fundamental period a slight perturbation of q complete periods of $x_0(t)$. Since the least period of $f((p/q)\cdot(t - \tau), x, \epsilon)$ is $2\pi q/p$, the solution is a subharmonic of order p.

3. The locking-in regions. We shall now study the regions of the ϵ-ω plane within which locking-in occurs. We apply Theorem A to the case that ω varies with ϵ. The general idea is the following. A rational p/q is selected, and we consider the line in the ϵ-ω plane given by

$$\omega = \frac{p}{q} + \eta\epsilon. \tag{3.1}$$

It will be found that for η sufficiently small, (but the range can be definitely described), there will be a range of ϵ for which locking-in occurs, while for η too large, no locking-in will occur on the line (3.1). The lines on which locking-in can occur generate a sector with vertex at $(0, p/q)$, and all locking-in related to the rational p/q occurs within this sector. Naturally the entire sector is not in the locking-in region, since ϵ must be small, but there is a small portion of the sector near the vertex for which locking-in occurs. This we call the locking-in region. Our analysis will determine the size of the sector within which the locking-in can occur. A similar analysis appears in [2], in which other aspects of locking-in are discussed.

The differential equation to be studied is thus

$$x' = g(x) + \epsilon f((p/q + \eta\epsilon)(t - \tau), x, \epsilon). \tag{3.2}$$

We make the change of independent variable

$$s = (1 + q\eta\epsilon/p)t.$$

With this change of variable, (3.2) becomes

$$(1 + q\eta\epsilon/p)\frac{dx}{ds} = g(x) + \epsilon f(p(s - \tau)/q - \eta\epsilon\tau, x, \epsilon).$$

Rearranging terms according to powers of ϵ this reduces to

$$\frac{dx}{ds} = g(x) + \epsilon\tilde{f}\left(\frac{p}{q}(s - \tau), x, \epsilon\right)$$

where

$$\epsilon\tilde{f}(p(s - \tau)/q, x, \epsilon) = \epsilon[f(p(s - \tau)/q, x, 0) - q\eta g(x)/p] + o(\epsilon).$$

Now equation (3.2) has the form to which Theorem A is applicable. Using Theorem A we have

THEOREM 1. *The equation*

$$\frac{dx}{ds} = g(x) + \epsilon\tilde{f}(p(s - \tau)/q, x, \epsilon) \tag{3.3}$$

has a periodic solution of period $2\pi q$ in s for sufficiently small ϵ if τ is a simple zero of

$$\tilde{F}_1(\tau) = \int_0^{2\pi q} z_0^*(s)\tilde{f}\left(\frac{p}{q}(s - \tau), x_0(s), 0\right) ds \tag{3.4}$$

$$= F_1(\tau) - (2\pi q^2/p)\eta.$$

Thus equation (3.2) will have a solution of period $2\pi q/(1 + (q\eta\epsilon/p))$ of the form

$$x(t) = x_0((1 + q\eta\epsilon/p)t) + \epsilon x_1((1 + q\eta\epsilon/p)t) + o(\epsilon)$$

if τ is a simple zero of $\tilde{F}_1(\tau)$. This solution is asymptotically stable if and only if

$$\epsilon\tilde{F}_1'(\tau) = \epsilon F_1'(\tau) < 0.$$

PROOF OF THEOREM 1. Except for the formula (3.4) for $\tilde{F}_1(\tau)$, the results follow directly from Theorem A and from the change of independent variable made in the differential equation. It remains only to prove the second formula for $\tilde{F}_1(\tau)$ in (3.4).

Since

$$\tilde{f}\left(\frac{p}{q}(s-\tau), x_0(s), 0\right) = f\left(\frac{p}{q}(s-\tau), x_0(s), 0\right) - \frac{q\eta}{p} g(x_0(s)),$$

we have

$$\tilde{F}_1(\tau) = \int_0^{2\pi q} z_0^*(s) f\left(\frac{p}{q}(s-\tau), x_0(s), 0\right) ds - \frac{q\eta}{p} \int_0^{2\pi q} z_0^*(s) g(x_0(s))\, ds$$
$$= F_1(\tau) - \frac{q\eta}{p} \int_0^{2\pi q} z_0^*(s) x_0'(s)\, ds,$$

this last following from the fact that $x_0(t)$ is a solution of (2.1). But by the definition of $z_0(t)$, $z_0^*(s)x_0'(s) \equiv 1$, so that

$$\tilde{F}_1(\tau) = F_1(\tau) - 2\pi q^2 \eta/p.$$

This completes the proof.

Since the requirement that $\tilde{F}_1(\tau) = 0$ is equivalent to

$$\eta = \frac{p}{2\pi q^2} F_1(\tau),$$

we see that η is restricted, and the sector within which locking-in occurs is bounded by the lines

$$\omega = \frac{p}{q} + \left[\frac{p}{2\pi q^2} \min F_1(\tau)\right]\epsilon$$

and

$$\omega = \frac{p}{q} + \left[\frac{p}{2\pi q^2} \max F_1(\tau)\right]\epsilon.$$

Since the function $F_1(\tau)$ depends on the choice of the rational p/q, the size of the sector will vary for different rationals. In cases for which $F_1(\tau)$ is identically constant, no sector will be determined, and our study will not predict the existence of a locking-in region.

Before we can say more about the locking-in regions and about the phase shift as a locking-in region is traversed, we must examine the structure of the function $F_1(\tau)$. We shall show that $F_1(\tau)$ has period $2\pi/p$, and that it reflects some of the properties of $f(\theta, x, \epsilon)$ considered as a function of θ. This will be done by constructing the Fourier series for $F_1(\tau)$.

Let $f(\theta, x, \epsilon)$ have the Fourier series

$$f(\theta, x, \epsilon) = \sum_{-\infty}^{\infty} A_m(x, \epsilon) \exp[im\theta]. \tag{3.5}$$

Then using this in the definition of $F_1(\tau)$ we have

$$F_1(\tau) = \int_0^{2\pi q} z_0^*(s) \sum_{-\infty}^{\infty} A_m(x_0(s), 0) \exp\left[im\frac{p}{q}(s-\tau)\right] ds. \tag{3.6}$$

Now the integrand in (3.6) has period $2\pi q$, but the function $z_0^*(s)A_m(x_0(s), 0)$ has period 2π. Thus by the properties of trigonometric series, the integrals

$$\int_0^{2\pi q} z_0^*(s) A_m(x_0(s), 0) \exp\left[im \frac{p}{q} (s - \tau) \right] ds$$

will vanish unless m is a multiple of q. Therefore we may rewrite (3.6) in the form

$$F_1(\tau) = \sum_{-\infty}^{\infty} \int_0^{2\pi q} z_0^*(s) A_{nq}(x_0(s), 0) \exp [inp(s - \tau)] \, ds \tag{3.7}$$

$$= \sum_{-\infty}^{\infty} a_n \exp [-inp\tau],$$

where

$$a_n = \int_0^{2\pi q} z_0^*(s) A_{nq}(x_0(s), 0) \exp [inps] \, ds$$

$$= q \int_0^{2\pi} z_0^*(s) A_{nq}(x_0(s), 0) \exp [inps] \, ds.$$

(We have assumed sufficient regularity in the differential equations so that the formal manipulation with Fourier series is justified.)

We can now make some observations about the structure of $F_1(\tau)$.

THEOREM 2.

1. *The function $F_1(\tau)$ has period $2\pi/p$.*

2. *If $f(\theta, x, \epsilon)$ is odd-harmonic in θ, then $F_1(\tau)$ is odd-harmonic if q is odd, and vanishes identically if q is even.*

3. *If $f(\theta, x, \epsilon)$ is a trigonometric polynomial in θ of degree N, then $F_1(\tau)$ is identically constant for $q > N$, and is a trigonometric polynomial of degree at most N/q if $q \leqq N$.*

4. *If $f(\theta, x, \epsilon)$ is a simple sinusoid in θ, $F_1(\tau)$ is a simple sinusoid when $q = 1$ and vanishes identically otherwise.*

PROOF OF THEOREM 2.

1. That $F_1(\tau)$ has period $2\pi/p$ follows directly from the series (3.7).

2. If $f(\theta, x, \epsilon)$ is odd-harmonic, then $A_m = 0$ if m is even. Thus $A_{nq} = 0$ unless n and q are both odd. Hence if q is even, every A_{nq} is zero, so that every $a_n = 0$. If q is odd $A_{nq} = 0$ if n is even, so that $a_n = 0$ when n is even. Thus in this case $F_1(\tau)$ is odd-harmonic.

3. If $f(\theta, x, \epsilon)$ is a polynomial, then $A_m = 0$ if $|m| > N$. If $q > N$, $A_{nq} = 0$ unless $n = 0$, so that $F_1(\tau) \equiv a_0$. If $q \leqq N$, only a finite number of the A_{nq} will be nonzero, and if $|nq| > N$, $A_{nq} = 0$. Thus $a_n = 0$ if $|nq| > N$, which shows that $F_1(\tau)$ is a polynomial of degree at most N/q.

4. If $f(\theta, x, \epsilon)$ is a sinusoid, then only A_1 and A_{-1} are nonzero. Thus if $q \neq 1$, $A_{nq} = 0$ for every n, so that $F_1(\tau)$ vanishes identically. If $q = 1$, $A_{nq} = 0$ unless $n = \pm 1$, so that $a_n = 0$ unless $n = \pm 1$. Thus $F_1(\tau)$ is a simple sinusoid of period $2\pi/p$.

This completes the proof.

It should be remarked here that locking-in is not impossible when $F_1(\tau) \equiv 0$. It is just that a deeper investigation involving perturbation terms with higher powers of ϵ is required.

4. Results on phase shift. By phase shift across a locking-in region we mean the change in τ corresponding to the stable locked-in solution as the parameter η goes from its minimum value to its maximum value. The cases of positive and negative ϵ are different but similar, and we shall confine our discussion to the case that ϵ is positive. We shall then have $F_1'(\tau) < 0$ for the stable solution.

If in the course of a period $(2\pi/p)$, $F_1(\tau)$ has several maxima and minima, there will be several τ-intervals which will correspond to stable locked-in solutions, so that it is not proper to speak of phase shift across the entire locking-in region. Rather we should speak of several different phase shifts corresponding to the several τ-intervals in which $F_1(\tau)$ decreases. Because of this complication, we shall confine our analysis of phase shift to the case where in the period $2\pi/p$, $F_1(\tau)$ has a single maximum and a single minimum.

The length of the τ-interval, which is usually a fraction of $2\pi/p$ is the phase shift with respect to the 2π-periodic function $x_0(t)$. To describe the phase shift with respect to the variable $\theta = pt/q$, such a phase shift must be multiplied by p/q.

THEOREM 3. *Let the function $F_1(\tau)$ have a single maximum and a single minimum in each period, and let the time from the maximum to the minimum be $\Delta\tau$. Let ϵ be positive. Then as the locking-in region is traversed, the phase shift τ decreases by $\Delta\tau$ and the corresponding phase shift in θ decreases by $p\Delta\tau/q$.*

PROOF OF THEOREM 3. Since the maximum and minimum values of η correspond to the maximum and minimum values of $F_1(\tau)$, and since $F_1'(\tau)$ must be negative for the stable solution, τ must decrease as η increases and the decrease in τ as we go from the minimum to the maximum of $F_1(\tau)$ is exactly $\Delta\tau$.

COROLLARY. *Suppose that $f(\theta, x, \epsilon)$ is odd-harmonic. Then $\Delta\tau = \pi/p$ and the shift in θ is π/q.*

When $f(\theta, x, \epsilon)$ is a simple sinusoid, we again have that f is odd-harmonic, so that $\Delta\tau = \pi/p$. In this case we must have $q = 1$, so that the shift in θ is always π when $f(\theta, x, \epsilon)$ is a sinusoid.

5. Conclusion. The motivation for this study arose in a conversation with Professor Karl Klotter. It had been conjectured that the phase shift across the principal locking-in region ($p = q = 1$) was π, and this study is an attempt to verify the conjecture and to generalize the result somewhat.

The results of this paper include the case of a second-order scalar system, where the unforced system is an equation of the van der Pol type with an orbitally stable limit cycle.

REFERENCES

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