ON A CLASS OF INTEGRAL EQUATION

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1. Introduction. The paper considers integral equations of the form

$$ag(x) = bf(x) - \lambda \int_0^\infty K(|x - y|)g(y)\, dy,\ x \geq 0, \tag{1}$$

where either a or b may be chosen to be zero. The method whilst different from Thielman's [1], applies to the same type of kernel, viz.

$$K(u) = \sum_{i=1}^{n} P_i(u) \exp(-k_i u) = \sum_{i=1}^{n} \sum_{j=0}^{m} a_{ij} u^j \exp(-k_i u),\ u \geq 0,\ k_i > 0, \tag{2}$$

i.e. the $P_i(u)$ are polynomials.

Thielman uses an ingenious elementary method ascribed to Lalesco [2] for solving this type of equation. Of course, the solution can also be obtained by a straightforward application of the Weiner–Hopf technique [3], but in the method outlined here elementary use is made of the Laplace transform.

2. Convolution theorem and formal solution. Let

$$\mathfrak{L}f \equiv \bar{f}(p) = \int_0^\infty \exp(-px)f(x)\, dx$$

denote the Laplace transform of $f(x)$ and consider

$$\begin{aligned} \mathfrak{L} \int_x^\infty K(y - x)g(y)\, dy &= \int_0^\infty \exp(-px)\, dx \int_x^\infty K(y - x)g(y)\, dy \\ &= \int_0^\infty g(y)\, dy \int_{-\infty}^{y} K(y - x) \exp(-px)\, dx - \int_0^\infty g(y)\, dy \int_{-\infty}^{0} K(y - x) \exp(-px)\, dx \\ &= \bar{K}(-p)\bar{g}(p) - \int_0^\infty g(y)\, dy \int_0^\infty K(y + x) \exp(px)\, dx \end{aligned} \tag{3}$$

provided all integrals are convergent. For the type of kernel under consideration

$$\begin{aligned} \int_0^\infty K(y + x) \exp(px)\, dx &= \sum_{i=1}^{n} \sum_{j=0}^{m} \int_0^\infty a_{ij}(y + x)^j \exp\{px - k_i(y + x)\}\, dx \\ &= \sum_{i=1}^{n} \sum_{j=0}^{m} \sum_{h=1}^{j+1} \frac{A_{ijh}}{(p - k_i)^h}\, y^{j+1-h} \exp(-k_i y) \end{aligned} \tag{4}$$

where, assuming all the integrals converge

$$A_{ijh} = \frac{(-1)^h j!}{(j + 1 - h)!}\, a_{ij}\ . \tag{5}$$

Then, substitution in (3) yields

$$\bar{K}(-p)\bar{g}(p) = \mathfrak{L} \int_x^\infty K(y - x)g(y)\, dy + \sum_{i=1}^{n} \sum_{j=0}^{m} \sum_{h=1}^{j+1} \gamma_{ijh}(p - k_i)^{-h} \tag{6}$$

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where

$$\gamma_{ijh} = A_{ijh} \int_0^\infty \exp(-k_i y) y^{j+1-h} g(y)\, dy \tag{7}$$

are finite constants if $g(y)$ is assumed to be such that the integrals (7) are also convergent. In fact this property of convergence of these integrals is required for the determination of the γ's as is shown in the two following examples.

The method proceeds as follows. Write equation (1) in the form

$$ag(x) = bf(x) - \lambda \int_0^x K(x-y)g(y)\, dy - \lambda \int_x^\infty K(y-x)g(y)\, dy$$

and then take its Laplace transform to give, on using Eq. (6)

$$a\bar{g}(p) = b\bar{f}(p) - \lambda\bar{K}(p)\bar{g}(p) - \lambda\bar{K}(-p)\bar{g}(p) + \sum_{i,j,h} \gamma_{ijh}(p-k_i)^{-h}. \tag{8}$$

Solve for $\bar{g}(p)$ and use the inversion theorem to find $g(x)$, and determine the γ's from convergency conditions (see example 1).

3. Illustrations of the method. The solution of problems where the kernel is complicated adds little to the understanding of the method. Consequently two simple examples are solved, an integral equation of the first kind with its associated homogeneous integral equation and one of the second kind, both with single term kernels.

EXAMPLE 1. Solve

$$f(x) = \lambda \int_0^\infty \exp\{-k|x-y|\}g(y)\, dy, \qquad k > 0.$$

Here

$$K(u) = \exp(-ku), \qquad \bar{K}(p) + \bar{K}(-p) = 2k/(k^2 - p^2);$$

$$i = 1, \quad j = 0, \quad h = 1, \quad a_{ij} = 1, \quad \gamma_{ijh} = \gamma \quad \text{say.}$$

From (8)

$$2k\lambda\bar{g}(p) = (k^2 - p^2)\bar{f}(p) - \gamma(k+p) \tag{9}$$

and this is written as

$$2k\lambda\bar{g}(p) = k^2\bar{f}(p) - \{p^2\bar{f}(p) - pf(0) - f^{(1)}(0)\} - [p\{\gamma + f(0)\} + k\gamma + f^{(1)}(0)]. \tag{10}$$

The expression in square brackets on the r.h.s. of Eq. (10) has a convergent inverse only when it is identically zero, i.e. when

$$\gamma = -f(0)$$

and

$$0 = k\gamma + f^{(1)}(0) \equiv -kf(0) + f^{(1)}(0). \tag{11}$$

The required solution is therefore

$$g(x) = \frac{k^2 f(x) - f^{(2)}(x)}{2k\lambda}$$

provided $f(x)$ satisfies the condition of Eq. (11).

If $f(x)$ is replaced by $g(x)$, the corresponding homogeneous equation is produced and then $g(x)$ is that solution of the differential equation

$$g^{(2)}(x) + (2\lambda - k)kg(x) = 0$$

which satisfies Eq. (11), viz. $g^{(1)}(0) - kg(0) = 0$. This can be verified by direct differentiation of the homogeneous integral equation

EXAMPLE 2. Solve

$$g(x) = f(x) - \lambda \int_0^\infty |x - y| \exp\{-k|x - y|\}g(y)\,dy, \qquad k > 0. \tag{12}$$

Here

$$K(u) = u \exp(-ku), \qquad \bar{K}(p) + \bar{K}(-p) = 2(k^2 + p^2)/(k^2 - p^2)^2,$$

$i = 1$, $j = 1$, $h = 1, 2$, so that Eq. (8) gives

$$\bar{g}(p) = \bar{f}(p) - \frac{2\lambda(k^2 + p^2)}{(k^2 - p^2)^2}\bar{g}(p) + \frac{\gamma_1}{p - k} + \frac{\gamma_2}{(p - k)^2}$$

which simplifies to

$$\bar{g}(p) = \left\{\bar{f}(p) + \frac{\gamma_1}{p - k} + \frac{\gamma_2}{(p - k)^2}\right\}\{1 - \Psi(p)\} \tag{13}$$

where

$$\Psi(p) \equiv \frac{2\lambda(k^2 + p^2)}{(k^2 - p^2)^2 + 2\lambda(p^2 + k^2)}.$$

Inversion of Eq. (13) yields, on using the usual convolution theorem,

$$g(x) = f(x) + (\gamma_1 + x\gamma_2)\exp(kx) - \int_0^x \{f(y) + (\gamma_1 + y\gamma_2)\exp(ky)\}\psi(x - y)\,dy$$

where γ_1, γ_2 are arbitrary, since all the integrals involved in the inversion of Eq. (13) are convergent.

REFERENCES

[1] H. P. Thielman, *On a class of singular integral equations occurring in physics*, Quart. Appl. Math. 6, 443 (1949)

[2] T. Lalesco, *Théorie des équations intégrales*, A. Hermann et fils, Paris 1912, p. 121

[3] N. Wiener and E. Hopf, *Über eine Klasse singularer Integralgleichungen*, Sitzungsber. Preuss. Ak. Wissensch. 1931, p. 696