SOLAR RADIANT HEATING OF A ROTATING SOLID CYLINDER*

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In an earlier paper, Olmstead and Raynor (1) have considered the temperature distribution in a solid cylinder exposed to solar radiation. They obtained a solution using the method of Green's functions and numerical examples were presented utilizing asymptotic expansions of the modified Bessel functions appearing in the formal solution.

In the present note, a Fourier series expansion of the radiation input function is used in conjunction with a simple product solution to obtain the results in a straightforward manner.

Using the notation of [1], the governing equation, in a coordinate system fixed relative to the incident radiation is

$$\frac{\partial^2 \tilde{T}}{\partial s^2} + \frac{1}{s}\frac{\partial \tilde{T}}{\partial s} + \frac{1}{s^2}\frac{\partial^2 \tilde{T}}{\partial \theta^2} + \zeta_0 \frac{\partial \tilde{T}}{\partial \theta} = 0 \tag{1}$$

with the boundary condition on the outside surface.

$$\frac{\partial \tilde{T}}{\partial s} + \beta \tilde{T} = -\frac{\beta}{4} + \frac{\pi}{4}\beta \cos\theta\, H(\cos\theta). \tag{2}$$

Here $\tilde{T}$ is the dimensionless temperature fluctuation about an average temperature T_0, s is the dimensionless radius and ζ_0 is the dimensionless angular velocity. β is defined as a dimensionless parameter $\beta = (4b\sigma e T_0^3/k)$ and $H(\cos\theta)$ is the Heaviside step function defined by

$$\begin{aligned} H(\cos\theta) &= 1, \quad \text{for} \quad 0 \le \theta \le \pi, \\ &= 0, \quad \text{for} \quad \pi \le \theta \le 2\pi. \end{aligned}$$

First expand the solar radiation input, $\cos\theta H(\cos\theta)$, in a Fourier series.

$$\cos\theta\, H(\cos\theta) = \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta] + \tfrac{1}{2}A_0 .$$

After evaluating the coefficients this becomes

$$\cos\theta\, H(\cos\theta) = \frac{1}{\pi} + \tfrac{1}{2}\cos\theta + \sum_{n=2}^{\infty} \frac{2(-1)^{1+n/2}}{\pi(n^2-1)} \cos n\theta. \tag{3}$$

SOLUTION. Since the temperature distribution must be periodic, consider a solution of the form

$$\tilde{T} = Ce^{-in\theta}P(s) \tag{4}$$

where continuity requires that n be an integer.

Substituting into Eq. (1) yields

$$\frac{d^2P}{ds^2} + \frac{1}{s}\frac{dP}{ds} - \left(i\zeta_0 n + \frac{n^2}{s^2}\right)P = 0 \tag{5}$$

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which is a form of Bessel's equation having solutions

$$P(s) = C_0\,, \qquad P(s) = C_n I_n((in\zeta_0)^{1/2}s). \tag{6}$$

Here the condition of regularity at $s = 0$ has been used and I_n is the modified Bessel function.

Let $\lambda_n \equiv (in\zeta_0)^{1/2}$. Then our temperature distribution $\tilde{T}$ is given by

$$\tilde{T} = C_0 + \sum_{n=1}^{\infty} [\{C_n I_n(\lambda_n s) + D_n I_n(i\lambda_n s)\} \cos n\theta + i\{D_n I_n(i\lambda_n s) - C_n I_n(\lambda_n s)\} \sin n\theta].$$

Applying the boundary conditions at the surface and equating coefficients gives

$$C_n = \frac{Z_n}{2X_n}, \qquad D_n = \frac{Z_n}{2Y_n}, \qquad C_0 = 0 \tag{7}$$

where

$$\begin{aligned} X_n &= \lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n), \\ Y_n &= i\lambda_n I_n'(i\lambda_n) + \beta I_n(i\lambda_n), \\ Z_n &= \tfrac{1}{8}\pi\beta, \qquad & n &= 1, \\ &= \frac{\beta(-1)^{1+n/2}}{2(n^2 - 1)}, \qquad & n &= 2, 4, 6 \cdots, \\ &= 0, \qquad & n &= 3, 5, 7, \cdots. \end{aligned} \tag{8}$$

Thus

$$\tilde{T} = \sum_{n=1}^{\infty} \left[\frac{Z_n I_n(\lambda_n s) e^{-in\theta}}{2\{\lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n)\}} + \frac{Z_n I_n(i\lambda_n s) e^{in\theta}}{2\{i\lambda_n I_n'(i\lambda_n) + \beta I_n(i\lambda_n)\}}\right]. \tag{9}$$

Since the coefficients in the summation are the complex conjugate of each other this may be written

$$\tilde{T} = \sum_{n=1}^{\infty} \left\{Z_n \operatorname{Re}\left[\frac{I_n(\lambda_n s)}{\lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n)}\right] \cos n\theta + Z_n \operatorname{Im}\left[\frac{I_n(\lambda_n s)}{\lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n)}\right] \sin n\theta\right\}. \tag{10}$$

Then using $T = T_0(1 + \tilde{T})$, the total temperature can be obtained. This yields

$$\frac{T}{T_0} = 1 + \beta\left\{\frac{\pi}{8}[a_1 \cos\theta + b_1 \sin\theta] + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2 - 1)} [a_{2n} \cos 2n\theta + b_{2n} \sin 2n\theta]\right\} \tag{11}$$

where

$$a_n \equiv \operatorname{Re}\left\{\frac{I_n(\lambda_n s)}{\lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n)}\right\},$$

$$b_n \equiv \operatorname{Im}\left\{\frac{I_n(\lambda_n s)}{\lambda_n I_n'(\lambda_n) + \beta I_n(\lambda_n)}\right\}$$

This solution is identical to that obtained by Olmstead and Raynor [1] through the more laborious and elegant Green function approach. Rather than expressing this result in terms of ber_n and bei_n functions and performing asymptotic expansions, numerical examples can be obtained utilizing a digital computer program for calculating the modified Bessel functions.

Reference

[1] W. E. Olmstead and S. Raynor, *Solar heating of a rotating solid cylinder*, Quart. Appl. Math. **21**, 81 (1963)