ON CIRCULARLY-POLARIZED NONLINEAR ELECTROMAGNETIC WAVES*

By M. M. CARROLL (*University of California, Berkeley*)

Summary. An exact solution of the nonlinear electromagnetic equations for a conservative nondispersive isotropic centro-symmetric dielectric is obtained, which has an especially simple form, i.e., a harmonic, circularly-polarized plane progressive wave. The normal incidence of such a wave at a plane interface is considered and also the propagation of such waves in a transversely isotropic, uniformly magnetized material.

1. Introduction. The classical electromagnetic theory is obviously inadequate for the study of the propagation of intense electromagnetic waves, such as are produced by lasers. Instead, a theory is needed in which the constitutive equations are nonlinear. Substitution from such nonlinear constitutive relations in Maxwell's equations gives a quasi-linear system of partial differential equations, which must be hyperbolic if electromagnetic waves are to propagate. Exact analysis of this system of equations, using the method of characteristics (see, e.g., Courant and Hilbert [1]), is rather complicated. Broer [2] obtained an exact solution for the reflection of a linearly-polarized wave incident normally, from vacuum, on the plane interface of a half-space occupied by a nondispersive isotropic nonlinear dielectric. An approximate theory, based on an assumption of small nonlinearity, has been developed extensively by Bloembergen and his co-workers. This theory, which applies also to dispersive and to anisotropic materials, is presented and relevant papers are reprinted in [3].

In the present paper an exact steady-state solution is obtained, which is valid for any conservative nondispersive isotropic centro-symmetric dielectric and which has an especially simple form, i.e., a harmonic circularly-polarized plane progressive wave. The phase velocity depends on the amplitude of the wave. The reflection and transmission of such a wave, incident normally, from vacuum, on the plane interface of a dielectric half-space, is considered. It is shown also that such waves can propagate in the symmetry direction of any conservative, transversely isotropic, centro-symmetric dielectric, to which a uniform static magnetic field may be applied in the symmetry direction.

2. Constitutive Equations. The electric vector $\mathbf{E}$ and the magnetic intensity vector $\mathbf{H}$ in a dielectric are assumed to be functions of the electric displacement vector $\mathbf{D}$ and magnetic induction vector $\mathbf{B}$, thus

$$\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \qquad \mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B}). \tag{2.1}$$

This assumption implies that the material is nondispersive. The symmetry of the material imposes restrictions on the form of the function dependence (2.1). For a homogeneous isotropic material possessing a center of symmetry, (2.1) has the form [4], [5]

$$\begin{aligned} \mathbf{E} &= \alpha_1\mathbf{D} + \alpha_2\mathbf{D} \times \mathbf{B} + \alpha_3(\mathbf{D}\cdot\mathbf{B})\mathbf{B}, \\ \mathbf{H} &= \beta_1\mathbf{B} + \beta_2(\mathbf{D}\cdot\mathbf{B})\mathbf{D} + \beta_3(\mathbf{D}\cdot\mathbf{B})\mathbf{D} \times \mathbf{B}, \end{aligned} \tag{2.2}$$

where the constitutive functions α_i and β_i ($i = 1, 2, 3$) are functions of the isotropic invariants

$$I_1 = \mathbf{D}\cdot\mathbf{D}, \qquad I_2 = \mathbf{B}\cdot\mathbf{B}, \qquad I_3 = (\mathbf{D}\cdot\mathbf{B})^2, \tag{2.3}$$

*Received August 1, 1966; revised manuscript received September 23, 1966.

thus

$$\alpha_i = \alpha_i(I_1, I_2, I_3), \qquad \beta_i = \beta_i(I_1, I_2, I_3). \tag{2.4}$$

The constitutive equations in a vacuum, for all field strengths, are

$$\mathbf{E} = \mathbf{D}, \qquad \mathbf{H} = \mathbf{B}. \tag{2.5}$$

For sufficiently weak fields (or for linear materials) Eqs. (2.2) have the familiar linearized form

$$\mathbf{E} = \alpha_0 \mathbf{D}, \qquad \mathbf{H} = \beta_0 \mathbf{B}. \tag{2.6}$$

For strong fields, however, the nonlinear terms in (2.2) must be taken into account.

It was shown in [5] that if the system is conservative, so that an electromagnetic energy density function $W(\mathbf{D}, \mathbf{B})$ exists, then

$$\mathbf{E} = \frac{\partial W}{\partial \mathbf{D}}, \qquad \mathbf{H} = \frac{\partial W}{\partial \mathbf{B}}. \tag{2.7}$$

Since W must depend on $\mathbf{D}$ and $\mathbf{B}$ through the isotropic invariants (2.3), comparison of (2.2) and (2.7) gives

$$\begin{aligned} \alpha_1 &= 2\frac{\partial W}{\partial I_1}, \qquad \alpha_2 = 0, \qquad \alpha_3 = 2\frac{\partial W}{\partial I_3}, \\ \beta_1 &= 2\frac{\partial W}{\partial I_2}, \qquad \beta_2 = 2\frac{\partial W}{\partial I_3}, \qquad \beta_3 = 0. \end{aligned} \tag{2.8}$$

If the material is nonmagnetic, then the constitutive equations (2.2) have the form[1]

$$\mathbf{E} = \alpha_1(I_1)\mathbf{D}, \qquad \mathbf{H} = \mathbf{B}. \tag{2.9}$$

3. Propagation of a finite amplitude plane wave. Consider the propagation of a finite amplitude plane wave in a conservative isotropic dielectric with constitutive equations (2.2), (2.8). A rectangular Cartesian coordinate system x is chosen, with the x_3-axis in the direction of propagation of the wave. Components in the system x are denoted by lower-case Latin or Greek subscripts, which take the values 1, 2, 3 and 1, 2, respectively, and the usual summation convention is adopted.

Maxwell's equations, in the absence of free currents and charges, are

$$\begin{aligned} \partial\mathbf{B}/\partial t + \operatorname{curl}\mathbf{E} &= 0, \qquad \operatorname{div}\mathbf{B} = 0, \\ \partial\mathbf{D}/\partial t - \operatorname{curl}\mathbf{H} &= 0, \qquad \operatorname{div}\mathbf{D} = 0. \end{aligned} \tag{3.1}$$

For a plane wave propagating in the x_3-direction, (3.1) gives

$$\frac{\partial B_\alpha}{\partial t} + \epsilon_{\alpha 3\beta}\frac{\partial E_\beta}{\partial x_3} = 0, \qquad \frac{\partial D_\alpha}{\partial t} - \epsilon_{\alpha 3\beta}\frac{\partial H_\beta}{\partial x_3} = 0, \tag{3.2}$$

where ϵ_{ijk} denotes the alternating symbol. Furthermore, the longitudinal components D_3 and B_3 are independent of x_3 and t and, for the present, initial or boundary conditions are assumed such that

$$D_3 = B_3 = 0. \tag{3.3}$$

[1]The constitutive equations for a nonmagnetic isotropic dielectric, which does not possess a center of symmetry, are also of the form (2.9).

The system of equations obtained by substitution for **E** and **H** from (2.2), (2.8) in (3.2) leads, in general, to rather complicated analysis. It admits, however, the following elementary solution.

Consider a circularly-polarized wave

$$\mathbf{D} = d(\cos\phi, \sin\phi, 0), \qquad \phi = kx_3 - \omega t, \tag{3.4}$$
$$\mathbf{B} = b(-\sin\phi, \cos\phi, 0),$$

where d, b, k and ω are real constants. The isotropic invariants (2.3) associated with the wave (3.4) are constants

$$I_1 = d^2, \qquad I_2 = b^2, \qquad I_3 = 0 \tag{3.5}$$

and the constitutive equations become

$$\mathbf{E} = \alpha\mathbf{D}, \qquad \mathbf{H} = \beta\mathbf{B}, \tag{3.6}$$

with

$$\alpha = \alpha_1(d^2, b^2, 0), \qquad \beta = \beta_1(d^2, b^2, 0). \tag{3.7}$$

The field equations (3.2) are satisfied, provided

$$\omega b = k\alpha d, \qquad \omega d = k\beta b, \tag{3.8}$$

so that the ratio b/d and the propagation velocity ω/k satisfy

$$b^2/d^2 = \alpha/\beta, \qquad \omega^2/k^2 = \alpha\beta. \tag{3.9}$$

For a nonmagnetic material, for example,

$$b^2 = \alpha_1(d^2)d^2, \qquad \omega^2/k^2 = \alpha_1(d^2). \tag{3.10}$$

Thus, a finite amplitude circularly-polarized plane progressive wave can propagate in any conservative nondispersive isotropic dielectric, with propagation velocity which depends on the wave amplitude.[2]

4. Reflection and transmission at a plane interface. Let the region $x_3 < 0$ be a vacuum and the region $x_3 > 0$ a dielectric with constitutive equations (2.2), (2.8) and consider a monochromatic circularly-polarized wave incident normally from vacuum on the interface $x_3 = 0$. The electric and magnetic intensity fields associated with the incident, reflected and transmitted waves are

$$\begin{aligned} E^i &= a^i(\cos\phi^i, \sin\phi^i, 0), & H^i &= a^i(-\sin\phi^i, \cos\phi^i, 0), \\ E^r &= a^r(-\cos\phi^r, \sin\phi^r, 0), & H^r &= a^r(\sin\phi^r, \cos\phi^r, 0), \\ E^t &= \alpha d(\cos\phi^t, \sin\phi^t, 0), & H^t &= \beta b(-\sin\phi^t, \cos\phi^t, 0), \end{aligned} \tag{4.1}$$

respectively, where

$$\phi^i = \omega(x_3 - t), \qquad \phi^r = \omega(x_3 + t), \qquad \phi^t = kx_3 - \omega t. \tag{4.2}$$

Here the amplitude a^i and the frequency ω (>0, say) of the incident wave are known, α and β are defined by (3.7) and b/d and ω/k (>0) are given by (3.9). The boundary conditions at the interface $x_3 = 0$ (continuity of the tangential components of **E** and **H**) give

$$a^i - a^r = \alpha d, \qquad a^i + a^r = \beta b. \tag{4.3}$$

---

[2]More generally, a finite amplitude plane wave $\mathbf{D} = d(\cos f(\phi), \sin f(\phi), 0)$, $\mathbf{B} = b(-\sin f(\phi), \cos f(\phi), 0)$, where $f(\phi)$ is an arbitrary function of ϕ, propagates without change of form in any conservative nondispersive isotropic dielectric, with phase velocity given by (3.9).

When Eqs. (3.9) are solved for b and k as functions of d, (4.3) is a system of two equations in the two unknowns d and a^r.[3] For a nonmagnetic material, for example, d is given by

$$\{1 + (\alpha_1(d^2))^{1/2}\}\, d(\alpha_1(d^2))^{1/2} = 2a^i \tag{4.4}$$

and a^r is given by

$$\{1 - (\alpha_1(d^2))^{1/2}\}\, d(\alpha_1(d^2))^{1/2} = 2a^r. \tag{4.5}$$

The fact that the transmitted wave is also monochromatic is surprising in a nonlinear theory.

5. Uniformly magnetized, transversely isotropic materials. Results similar to those obtained in Sections 3 and 4 hold in a more general situation. Consider a material which is transversely isotropic with respect to the x_3-direction, e.g., a uniaxial crystal. If an electromagnetic energy density $W(\mathbf{D}, \mathbf{B})$ exists, then the requirement that W be invariant with respect to any rotation of axes about the x_3-direction leads to

$$W = W(D_\alpha D_\alpha\ , B_\alpha B_\alpha\ , D_\alpha B_\alpha\ , D_3\ , B_3). \tag{5.1}$$

If the material possesses a center of symmetry, then W must also be invariant with respect to the central inversion transformation, for which (since $\mathbf{D}$ is an absolute vector and $\mathbf{B}$ is an axial vector)

$$D_i\ , B_i \to -D_i\ , B_i\ . \tag{5.2}$$

Thus

$$W = W(D_\alpha D_\alpha\ , B_\alpha B_\alpha\ , B_3\ , (D_\alpha B_\alpha)^2, D_3^2\ , D_3 D_\alpha B_\alpha),$$

or

$$W = W(I_1\ , I_2\ , I_3\ , I_4\ , I_5\ , I_6), \tag{5.3}$$

where I_1 , I_2 and I_3 are given by (2.3) and

$$I_4 = B_3\ , \qquad I_5 = D_3^2\ , \qquad I_6 = D_3 D_\alpha B_\alpha\ . \tag{5.4}$$

Then, from (2.7), the constitutive equations have the form

$$\begin{aligned} E_i &= 2\frac{\partial W}{\partial I_1} D_i + 2\frac{\partial W}{\partial I_3} D_j B_j B_i + \left(2\frac{\partial W}{\partial I_5} D_3 + \frac{\partial W}{\partial I_6} D_\alpha B_\alpha\right)\delta_{3i} + \frac{\partial W}{\partial I_6} D_3 B_\alpha \delta_{\alpha i}\ , \\ H_i &= 2\frac{\partial W}{\partial I_2} B_i + 2\frac{\partial W}{\partial I_3} D_j B_j D_i + \frac{\partial W}{\partial I_4}\delta_{3i} + \frac{\partial W}{\partial I_6} D_3 D_\alpha \delta_{\alpha i}\ , \end{aligned} \tag{5.5}$$

where δ_{ij} denotes the Kronecker delta.

Consider the propagation of a circularly-polarized wave of the form (3.4) in such a material and suppose that uniform static electric and magnetic fields are also applied in the x_3-direction, so that the equations (3.4) become

$$\begin{aligned} \mathbf{D} &= (d\cos\phi,\ d\sin\phi,\ \mathfrak{D}), \\ \mathbf{B} &= (-b\sin\phi,\ b\cos\phi,\ \mathfrak{B}), \end{aligned} \qquad \phi = kx_3 - \omega t. \tag{5.6}$$

---

[3]Of course, the requirements of existence and uniqueness of solution impose restrictions on the form of the constitutive functions α_1 and β_1.

The invariants associated with the fields (5.6) are constants

$$
\begin{aligned}
I_1 &= d^2 + \mathfrak{D}^2, & I_4 &= \mathfrak{B}, \\
I_2 &= b^2 + \mathfrak{B}^2, & I_5 &= \mathfrak{D}^2, \\
I_3 &= \mathfrak{D}^2\mathfrak{B}^2, & I_6 &= 0.
\end{aligned} \tag{5.7}
$$

Thus, whenever $\mathfrak{D} = 0$, the constitutive equations (5.5) have the form

$$E_i = \alpha D_i\,, \qquad H_i = \beta B_i + \gamma \delta_{3i}\,, \tag{5.8}$$

where α, β and γ are constants

$$\alpha = 2\frac{\partial W}{\partial I_1}\,, \qquad \beta = 2\frac{\partial W}{\partial I_2}\,, \qquad \gamma = \frac{\partial W}{\partial I_4}. \tag{5.9}$$

Consequently, results similar to those obtained previously hold also in the case of propagation in the symmetry direction of a transversely isotropic dielectric, possessing a center of symmetry, to which a uniform static magnetic field may also be applied in the symmetry direction. Similar results are not obtained, in general, in the case of an applied uniform static electric field.[4]

Acknowledgment. The results reported here were obtained in the course of research supported by the U. S. Office of Naval Research under Contract Nonr 222(69) with the University of California, Berkeley.

References

[1] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962
[2] L. J. F. Broer, Physics Letters **4**, 65 (1963)
[3] N. Bloembergen, *Nonlinear optics*, W. A. Benjamin Inc., New York, 1965
[4] A. C. Pipkin and R. S. Rivlin, J. Math. Phys. **1**, 542 (1960)
[5] M. M. Carroll and R. S. Rivlin, Quart. Appl. Math. **23**, 365 (1966)

[4]Roughly speaking, a material which possesses a center of symmetry retains its centro-symmetric property in the presence of a uniform static magnetic field, but not in the presence of a uniform static electric field.