# DIRECTIVITY FOR ONE-DIMENSIONAL SCALAR RADIATION*

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**Abstract.** This is an elementary contribution to the theory of directivity for scalar radiation. An infinite string is made to vibrate by application of external force to a finite portion of it. It is shown how this force may be applied so that all the energy travels off to the right and none to the left.

It is of interest to look at the problem of directivity for scalar radiation in the simplest case, viz. the case of one dimension. As a completely representative model of this we may take an infinite string vibrating transversely. The displacement $z$ satisfies the equation

$$\partial^2 z/\partial t^2 - \partial^2 z/\partial x^2 = F \tag{1}$$

the units being chosen so that the speed of propagation is unity; here $F$ is the applied force.

Taking the string to be initially undisturbed ($z = \partial z/\partial t = 0$ for $t = 0$), the d'Alembert solution[1] is

$$z(x, t) = \frac{1}{2} \int_{T(x,t)} F(\xi, \tau)\, d\xi\, d\tau, \tag{2}$$

the integration being taken over the characteristic triangle $T(x, t)$, this triangle being bounded by the characteristics $dx/dt = \pm 1$ through the point $(x, t)$ and by the $x$-axis. Fig. 1 shows $T(x_1, t)$ and $T(x_2, t)$ in a space-time diagram, with $x_1$ to the left and $x_2$ to the right.

Let $F(x, t)$ be zero outside $0 < x < a$. To follow the disturbance $z(x_1, t)$ we draw the line $x = x_1$, and carry out the integration (2) over the succession of characteristic triangles $T(x_1, t)$ for $t > 0$. Likewise for $z(x_2, t)$.

Our aim is to direct the radiation to the right. A simple way to do this is to assign to $F$ the values $\pm 1$ as indicated in Fig. 1, with $F = 0$ elsewhere; thus $F = 1$ in the triangle with vertices $(0, 0)$, $(a, 0)$, $(a, a)$ and $F = -1$ in $(0, 0)$, $(a, a)$, $(0, a)$. On account of the skew-symmetry of this distribution about the diagonal drawn from $(0, 0)$ to $(a, a)$, it is clear from (2) that $z(x_1, t) = 0$ for $x_1 < 0$, and so there is no disturbance on the left. As for the disturbance on the right, at $x_2$ where $x_2 > a$, we find

$$\begin{aligned} z(x_2, t) &= 0 && \text{for} \quad t < x_2 - a \\ &= \tfrac{1}{4}(t - x_2 + a)^2 && \text{for} \quad (x_2 - a) < t < x_2, \\ &= \tfrac{1}{4}(t - x_2 - a)^2 && \text{for} \quad x_2 < t < (x_2 + a), \\ &= 0 && \text{for} \quad t > (x_2 + a). \end{aligned} \tag{3}$$

This represents a pulse travelling out to the right.

It is easy to describe how $F$ is to be applied physically. We start by pushing the string up with a uniform force in the part $(0, a)$, but immediately reverse the pressure

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[1]Cf. H. F. Weinberger, *Partial differential equations*, Blaisdell Publishing Co., New York, 1965, p. 26.

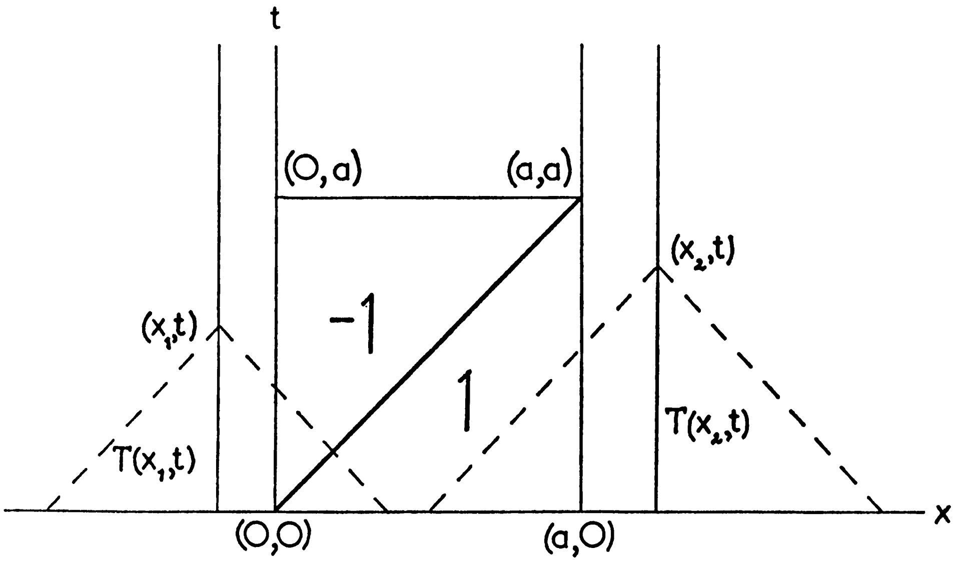


FIG. 1. Space-time diagram for vibrating string.

on the left, and carry this reversal to the right with unit speed until the reversal is complete. The pressure is then removed.

Although the desired effect can be produced by other choices of $F$, the above is perhaps the simplest. The essential property is that the integral of $F$ shall be zero when calculated for the domain lying under *any* characteristic $dx/dt = -1$. It is convenient therefore (although not necessary) to make $F$ skew-symmetric about the diagonal $x = t$.

Instead of taking $F$ as above, let us make it a distribution, constant along the diagonal from (0, 0) to $(a, a)$ and zero elsewhere; in terms of the Dirac delta-function $F(x, t) = \delta(x - t)$ for $0 < x < a$ with $F = 0$ elsewhere. This excitation corresponds to running a pressure point along the string with the speed of propagation (unity).

It is easy to see that a disturbance travels out in both directions, with a sharp step on the right and a sloping step on the left.

Let us modify this last excitation by taking two pressure points close to each other, one pressing up and the other down, both running along the string towards the right at unit speed. This corresponds to $F(x, t) = \delta'(x - t)$, and now we get a sharp wave on the right (a delta-function) and no disturbance at all on the left.

A common feature of the above choices of $F$ is that $F$ (or a discontinuity of $F$) travels to the right with the speed of propagation (unity), with a history in the space-time diagram represented by the heavy diagonal line.

Should we wish to excite the portion $(0, a)$ but have no leakage in either direction, we might divide the square shown in Fig. 1 into four equal triangles and assign $F = \pm 1$ in them so as to get skew-symmetry about each diagonal.

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