SHEAR WAVES IN FINITE ELASTIC STRAIN*

BY J. D. ACHENBACH (*Northwestern University*)

1. Introduction. In this note the analytical tools of the theory of propagating surfaces of discontinuity are used to construct the solution of a transient wave propagation problem. We consider wave motion in pure shear generated by the application of a monotonically increasing transverse particle velocity at the surface of a half-space that is initially undisturbed. The material of the half-space is nonlinear elastic, incompressible and isotropic.

The expression for the displacement which is obtained by considering propagating discontinuities shows that the magnitude of the acceleration discontinuity at the wave front remains constant in time. In another study of the present problem by means of the method of characteristics it was, however, pointed out by Chu [1] that the formation of shock waves should not be ruled out, because a preceding disturbance may be overtaken by subsequent disturbances. The latter effect, which for the present boundary conditions may occur if the stress-deformation relation is convex with respect to the deformation gradient axis, is not accounted for in studies of propagating discontinuities. The solution that is presented here is valid at all times if the stress-deformation relation is concave, and it is valid till the time of shock formation if the latter relation is convex.

2. Governing equations. For an elastic material that is isotropic and incompressible the constitutive equation is of the form, [2, Eq. (6.6)],

$$\sigma_{ij} = 2\left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)g_{ij} - g_{ik}g_{kj}\frac{\partial W}{\partial I_2}\right] - p\delta_{ij} . \tag{1}$$

In Eq. (1), $W(I_1, I_2)$ is the strain energy function, and p is an arbitrary hydrostatic pressure. The strain-invariants I_1 and I_2 are defined as

$$I_1 = g_{ii} , \qquad I_2 = \tfrac{1}{2}(g_{ii}g_{jj} - g_{ij}g_{ji}) \tag{2a, b}$$

where

$$g_{ij} = (\partial x_i/\partial X_m)(\partial x_j/\partial X_m). \tag{3}$$

Consider a half-space $X_2 \geq 0$ that is initially undisturbed. Let a translational motion in the X_1 direction be imparted to the half-space by application of a spatially uniform, but time-dependent particle velocity $v(t)$. The surface of the half-space is not allowed to move in the X_2 direction. As pointed out by Chu [1], the displacement field for $t > 0$ may be described by

$$x_1 = X_1 + \lambda(X_2 , t), \qquad x_2 = X_2 , \qquad x_3 = X_3 . \tag{4a, b, c}$$

Substitution of (4a, b, c) in Eqs. (3) and (2a, b) yields for this particular deformation

$$I_1 = I_2 = 3 + F^2 \tag{5}$$

in which

$$F = g_{12} = \partial\lambda(X_2 , t)/\partial X_2 . \tag{6}$$

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The shear stress σ_{12}, henceforth denoted by σ, is now obtained from Eq. (1) as

$$\sigma = \sigma_{12} = 2((\partial W/\partial I_1) + (\partial W/\partial I_2))F. \tag{7}$$

By assuming that W is a polynomial function of the strain invariants I_1 and I_2, it follows, in view of Eq. (5), that $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are functions of F^2 only. The stress-deformation relation (7) may then be represented by the expansion

$$\sigma = F \sum_{i=0}^{N} A_i F^{2i} \tag{8}$$

where A_i are constants.

The relevant equation of motion becomes simply

$$\rho(\partial^2\lambda/\partial t^2) = \partial\sigma/\partial X_2 . \tag{9}$$

We assume that the particle velocity at $X_2 = 0$ is applied gradually, such that in a Maclaurin expansion the first term vanishes, i.e.,

$$\frac{\partial\lambda(0, t)}{\partial t} = v(t) = \sum_{n=1}^{\infty} \frac{1}{n!} v_n t^n. \tag{10}$$

It is assumed that $dv/dt \geq 0$ at all times.

3. Series solution. The form of the boundary condition (10) suggests that propagating acceleration discontinuities as well as higher order discontinuities will be generated in the half-space. It is easy to show [3], [4] that transverse acceleration waves and transverse waves of higher orders, if considered separately as isolated discontinuities propagating into a previously undisturbed medium, move with velocities

$$c_0 = (A_0/\rho)^{1/2} \tag{11}$$

where A_0 is defined by (8). Since none of the conceivably generated discontinuities can move separately with a velocity larger than c_0 it can be stated, with a qualification, that the wave front of the total disturbance, which is generated at time $t = 0$ at $X_2 = 0$ by application of (10), also propagates with velocity c_0. The qualification is that the statement is correct if subsequent disturbances propagate with velocities smaller than c_0. If such subsequent disturbances overtake the wave front (shock formation) at time t^*, the statement is correct for $t < t^*$.

Since the material is undisturbed until the wave front arrives, we seek the displacement at an arbitrary position X_2 as a Taylor's expansion about the time of arrival of the wave front. Using a familiar notation for discontinuities, we write for fixed X_2 and for $t \geq X_2/c_0$:

$$\lambda(X_2 , t) = \sum_{n=2}^{\infty} \frac{1}{n!} (t - X_2/c_0)^n [\partial^n\lambda/\partial t^n]. \tag{12}$$

Basic to the study of the magnitudes of propagating discontinuities is the kinematical condition of compatibility. For a function $f(X_2 , t)$ which is discontinuous and has discontinuous derivatives across a surface that moves in the X_2 direction with velocity c_0, this condition takes the form

$$(d/dt)[f] = [\partial f/\partial t] + c_0[\partial f/\partial X_2]. \tag{13}$$

By applying (13) to the derivatives $\partial^{n-1}\sigma/\partial t^{n-1}$ and $\partial^n\lambda/\partial t^n$ for $n \geq 1$, we obtain

$$\frac{d}{dt}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] = \left[\frac{\partial^n \sigma}{\partial t^n}\right] + c_0\left[\frac{\partial^n \sigma}{\partial X_2\, \partial t^{n-1}}\right], \tag{14}$$

$$\frac{d}{dt}\left[\frac{\partial^n \lambda}{\partial t^n}\right] = \left[\frac{\partial^{n+1}\lambda}{\partial t^{n+1}}\right] + c_0\left[\frac{\partial^n F}{\partial t^n}\right]. \tag{15}$$

By employing (8) we obtain the following relation between discontinuities at the wave front

$$[\partial^n \sigma/\partial t^n] = A_0[\partial^n F/\partial t^n] + P_n(t) \tag{16}$$

where

$$P_n(t) = \left[\frac{\partial^n}{\partial t^n}\left\{F \sum_{i=1}^{N} A_i F^{2i}\right\}\right]. \tag{17}$$

Another relation between discontinuities at the wave front is obtained from the equation of motion (9)

$$[\partial^n \sigma/\partial X_2\, \partial t^{n-1}] = \rho[\partial^{n+1}\lambda/\partial t^{n+1}]. \tag{18}$$

We now form the sum (16) + c_0 (18), where c_0 is defined by (11). By employing the relations (14) and (15) this sum reduces to (for $n \geq 2$):

$$\frac{d}{dt}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] = c_0 \rho \frac{d}{dt}\left[\frac{\partial^n \lambda}{\partial t^n}\right] + P_n(t). \tag{19}$$

For an initially undisturbed material $P_n(t)$ does not contain discontinuities in time derivatives of F of orders higher than $n - 2$. By employing the kinematic condition of compatibility and the equation of motion, $(d/dt)[\partial^{n-1}\sigma/\partial t^{n-1}]$ can be eliminated from (19), and we obtain

$$2c_0\rho \frac{d}{dt}\left[\frac{\partial^n \lambda}{\partial t^n}\right] = \frac{d^2}{dt^2}\left[\frac{\partial^{n-2}\sigma}{\partial t^{n-2}}\right] - P_n(t). \tag{20}$$

The inhomogeneous differential equation (20) recursively yields the magnitudes of discontinuities ($n \geq 2$):

$$\left[\frac{\partial^n \lambda}{\partial t^n}\right] = \frac{1}{2c_0\rho}\int_0^t \frac{d^2}{dt^2}\left[\frac{\partial^{n-2}\sigma}{\partial t^{n-2}}\right] - \frac{1}{2c_0\rho}\int_0^t P_n(s)\, ds + v_{n-1} \tag{21}$$

where the v_{n-1} are defined by (10). The coefficients of the Taylor expansion (12) are obtained by replacing t by X_2/c_0 in Eq. (21). The first three coefficients are

$$\left[\frac{\partial^2 \lambda}{\partial t^2}\right] = v_1\,, \tag{22}$$

$$\left[\frac{\partial^3 \lambda}{\partial t^3}\right] = v_2 + 3(A_1 v_1^3/A_0 c_0^3)X_2\,, \tag{23}$$

$$\left[\frac{\partial^4 \lambda}{\partial t^4}\right] = v_3 + 18(A_1 v_1^2 v_2/A_0 c_0^3)X_2 + 27(A_1^2 v_1^5/A_0^2 c_0^6)X_2^2\,. \tag{24}$$

It is noted that $[\partial^2\lambda/\partial t^2]$ remains constant, while the other coefficients are functions of the distance from the free surface. The validity of (12) is discussed in the next section.

4. Comparison with solution by method of characteristics. By employing the method of characteristics the solution of the problem governed by Eqs. (8-10) was obtained by Chu [1] in the following form

$$\partial\lambda(X_2 , t)/\partial t = v(\alpha) \tag{25}$$

$$\int_0^F c_s(F^2)\, dF = -v(\alpha) \tag{26}$$

where $v(t) = \partial\lambda(0, t)/\partial t$ is defined by Eq. (10). In Eq. (26):

$$\{c_s(F^2)\}^2 = \frac{1}{\rho} \frac{d}{dF} \left\{F \sum_{i=0}^{N} A_i F^{2i}\right\}. \tag{27}$$

The characteristic variable α is defined by

$$d\alpha = 0 \quad \text{along} \quad dX_2/dt = c_s(F^2). \tag{28}$$

From Eq. (26) it is noted that c_s is a function of α, and since α is constant along a characteristic, it then follows from (28) that the characteristics are straight lines. If the characteristics are labeled such that $\alpha = t$ at $X_2 = 0$, Eq. (28) can be integrated, to yield

$$X_2 = c_s(\alpha)(t - \alpha). \tag{29}$$

To obtain an explicit solution for $\partial\lambda(X_2 , t)/\partial t$ in terms of X_2 and t, $c_s(\alpha)$ must be solved from (26), after which α must be expressed in terms of X_2 and t by using Eq. (29). It is apparent that it is, in general, not possible to obtain a simple closed-form expression for $\alpha(X_2 , t)$. It appears that here we must also resort to a series expansion. At X_2 a Taylor expansion of $v(\alpha)$ about $\alpha = 0$ (the wave front) is written as

$$v(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n!} \{t - X_2/c_s(0)\}^n \left.\frac{\partial^n v}{\partial t^n}\right|_{\alpha=0}. \tag{30}$$

By employing the chain-rule of differentiation, where $\partial^i\alpha/\partial t^i$ is obtained from (29), (26) and (27), the coefficients $\partial^n v/\partial t^n$ $(\alpha = 0)$ are obtained as functions of X_2. It can be checked that the coefficients are identical to those obtained in a much less cumbersome manner from Eq. (21).

In the discussion following Eq. (29) it was tacitly assumed that (29) can be solved for α in terms of X_2 and t. As pointed out, however, by Chu [1], such a solution is possible only if there is no positive X_2 for which

$$X_2 = c_s^2/(dc_s/d\alpha). \tag{31}$$

It is clear that (31) can be satisfied only if $dc_s/d\alpha > 0$, i.e., if a characteristic for $\alpha > \alpha^*$ intersects the characteristic at $\alpha = \alpha^*$, and a preceding disturbance is overtaken by a subsequent disturbance. From (26) we obtain

$$\frac{dc_s}{d\alpha} = \frac{dc_s}{d(F^2)} 2F \frac{dF}{d\alpha} = -2 \frac{F}{c_s(\alpha)} \frac{dv}{d\alpha} \frac{dc_s}{d(F^2)}. \tag{32}$$

Thus for a monotonically increasing particle velocity at $X_2 = 0$, i.e., $dv(\alpha)/d\alpha > 0$, we can have shock formation if $dc_s(F^2)/dF^2 > 0$ (convex stress-deformation curve). The Taylor expansions (12) and (30) are then valid only up to the time of shock formation, where an increasing number of terms must be used as this time is approached.

For $dc_s(F^2)/dF^2 < 0$ (concave stress-deformation curve) the equality (32) can never be satisfied for a monotonically increasing surface particle velocity, and the expansions (12) and (30) are valid at all times.

The method presented here can also be used if the material is viscoelastic [5].

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