THE INVARIANT IMBEDDING EQUATION FOR THE DISSIPATION FUNCTION OF A HOMOGENEOUS FINITE SLAB*

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I. Introduction. The physical situation to be considered in this note is as follows: parallel rays of radiation are incident on a finite homogeneous slab which absorbs and scatters radiation isotropically. The diffusely transmitted and reflected fields have been studied intensively. In earlier papers [1]-[3] we have shown the importance of the dissipation function in various analytical studies of transport in a rod. In this note we derive an equation for the dissipation function of a slab and write the conservation relation which relates the reflection, transmission, and dissipation functions.

II. Derivation of invariant imbedding equation for the absorption function. Consider a plane-parallel, homogeneous and isotropically scattering medium of finite optical thickness τ_1. Suppose that a parallel beam of radiation of constant net flux π per unit area normal to the incident direction is incident on the upper surface $\tau = 0$ at a fixed angle whose cosine is μ_0 $(0 < \mu_0 \leq 1)$ with respect to the inward normal. We follow the standard nomenclature of Chandrasekhar [4].

The intensity of radiation which is diffusely reflected from the slab with direction cosine μ is $S(\tau_1 ; \mu, \mu_0)/4\mu$, and the diffusely transmitted intensity with direction cosine μ is $T(\tau_1 ; \mu, \mu_0)/4\mu$. The directly transmitted flux is $\pi \exp(-\tau_1/\mu_0)$ in the direction of incidence.

We define the absorption function L in the following fashion. Let

$\pi L(\tau_1, \mu_0)$ = the rate of production of truly absorbed particles in a cylinder of unit base area extending from $\tau = 0$ to $\tau = \tau_1$, the input having direction cosine μ_0 and the net incident flux being π.

The probability of ultimate absorption of a particle which is incident on a slab of thickness τ_1 with direction cosine μ_0 is $L(\tau_1, \mu_0)/\mu_0$.

We add an infinitesimal layer of optical thickness Δ to the lower surface τ_1, and we consider its effect on the rate of production of absorbed particles. We obtain the equation

$$\pi L(\tau_1 + \Delta, \mu_0) = \pi L(\tau_1, \mu_0) + \left[\pi\mu_0 \exp(-\tau_1/\mu_0)\frac{\Delta}{\mu_0} + \int_0^1 \frac{T(\tau_1 ; \mu', \mu_0)}{4\mu'}\,\mu'\,\frac{\Delta}{\mu'}\,2\pi\,d\mu'\right] \cdot \left[(1-\lambda) + \frac{\lambda}{4\pi}\int_0^1 \frac{L(\tau_1, \mu_0')}{\mu_0'}\,2\pi\,d\mu_0'\right] + o(\Delta), \qquad (1)$$

where λ is the albedo for single scattering. The first term on the right-hand side of the equation accounts for the absorption of particles which never enter the thin slab. The second term accounts for those particles which interact in the thin slab and then are absorbed. The first bracketed expression represents the rate of production of interacting particles in the cylinder of unit base area extending from $\tau = \tau_1$ to $\tau = \tau_1 + \Delta$, and

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the second bracketed expression is the probability that an interacting particle is ultimately absorbed. All other processes have probabilities of order Δ^2 or greater and are accounted for in the term $o(\Delta)$. Letting $\Delta \to 0$, we obtain the partial differential integral equation

$$\frac{\partial L(\tau_1, \mu_0)}{\partial \tau_1} = \left[\exp(-\tau_1/\mu_0) + \frac{1}{2}\int_0^1 T(\tau_1; \mu', \mu_0)\frac{d\mu'}{\mu'}\right]\left[1 - \lambda + \frac{\lambda}{2}\int_0^1 L(\tau_1, \mu_0')\frac{d\mu_0'}{\mu_0'}\right]. \tag{2}$$

The initial condition is

$$L(0, \mu_0) = 0. \tag{3}$$

III. Conservation relationship. The particles incident on a unit of horizontal area are either directly transmitted, truly absorbed, diffusely reflected, or diffusely transmitted. This leads to the conservation relationship

$$1 = \exp(-\tau_1/\mu_0) + \frac{L(\tau_1, \mu_0)}{\mu_0} + \frac{1}{2\mu_0}\int_0^1 S(\tau_1, \mu', \mu_0)\, d\mu' + \frac{1}{2\mu_0}\int_0^1 T(\tau_1, \mu', \mu_0)\, d\mu'. \tag{4}$$

IV. Discussion. The differential-integral Eq. (2) for L and the differential-integral equations for S and T which are given in Chandrasekhar's book [4] together with their initial conditions, form a system of equations which can be integrated numerically using the method of finite ordinates [2].

We also wish to point out that it may be possible to establish an existence and uniqueness theorem for the differential-integral equation for S and T making use of the conservation relation given above and the nonnegativity of S, T and L. This program was successfully carried out for a closely related process in [1] and [3].

References

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