EQUATIONS EQUIVALENT TO A FIRST ORDER EQUATION UNDER DIFFERENTIATION*

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1. Introduction. In this note we discuss some solutions of the quasilinear second order partial differential equation

$$u_{tt} - (F_p/F_q)^2 u_{xx} + (F_u/F_q^2)(F_q u_t - F_p u_x) + (1/F_q^2)(F_q F_t - F_p F_x) = 0. \tag{1.1}$$

For this equation and throughout the sequel, subscripts shall denote differentiation with respect to the indicated variable. We assume that all solutions obtained have sufficient differentiability conditions imposed on them. Equation (1.1) is obtained by differentiation of the nonlinear first order equation

$$F(x, t, u, p, q) = 0 \tag{1.2}$$

where $p = u_x$, $q = u_t$. Equations of the form (1.1) arise in many areas of application. Some of these are discussed in Ames [1].

2. Basic results. We merely remark that if U satisfies (1.2) then, whenever U is twice differentiable, U satisfies (1.1). The following example will serve to introduce the method.

Example. Suppose that in (1.1) we set

$$(F_p/F_q)^2 = \phi^2(u) \tag{2.1}$$

where ϕ is a twice differentiable function. (2.1) leads to the two equations

$$F_p + \phi(u)F_q = 0 \tag{2.2}$$

and

$$F_p - \phi(u)F_q = 0. \tag{2.3}$$

The method will become apparent if the second of these two equations is considered. (2.3) is a first order partial differential equation for F. Assume that $F = F(u, p, q)$. Then by Lagrange's Method

$$F = q + \phi(u)p \tag{2.4}$$

is a solution to (2.3). The determination of (1.1) can now be completed with (2.4). This is

$$u_{tt} - \phi^2(u)u_{xx} - \phi(u)\phi'(u)u_x^2 + \phi'(u)u_x u_t = 0, \tag{2.5}$$

which can be written

$$u_{tt} - [\phi^2(u)u_x]_x + \phi'(u)u_x(u_t + \phi(u)u_x) = 0. \tag{2.6}$$

Integrating $F = 0$ in (2.4), we obtain

$$u = G(x - t\phi(u)) \tag{2.7}$$

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as a solution for arbitrary, G. By the remark at the beginning of this section, (2.7) is a solution of (2.6). Also, this means that (2.7) is a solution of

$$u_{tt} - [\phi^2(u)u_x]_x = 0. \tag{2.8}$$

Approaching (2.2) in the same way as (2.3), it is seen that

$$u = H(x + t\phi(u)) \tag{2.9}$$

is a solution to (2.8) for arbitrary H.

3. An application. Tomotika and Tamada [2] and Tamada [3] have considered the equation

$$(ku)_{tt} = [(ku)^2]_{xx} \tag{3.1}$$

in connection with the nearly uniform transonic flow of a real gas obeying the adiabatic law, $k = (\gamma + 1)/2$, where γ is the ratio of the specific heats. Tomotika and Tamada have investigated (3.1) by making several "ad hoc" assumptions about the form of the solution. (3.1) is a special case of (2.8). Without loss of generality, take $k = 1$. Then,

$$\phi^2(u)u_x = (u^2)_x = 2uu_x \tag{3.2}$$

will determine ϕ. Now, by (2.7) and (2.9) we have

$$u = G(x - t(2u)^{1/2}) \tag{3.3}$$

and

$$u = H(x + t(2u)^{1/2}) \tag{3.4}$$

as solutions to (3.1).

References

[1] W. F. Ames, *Nonlinear partial differential equations in engineering*, Academic Press, New York, 1965

[2] S. Tomotika and K. Tamada, *Studies on two-dimensional transonic flows of compressible fluid*, Quart. Appl. Math. 7, No. 4, 381–397 (1949)

[3] K. Tamada, *Studies on the two-dimensional flow of a gas, with special reference to the flow through various nozzles*, Ph.D. Thesis, University of Kyoto, Japan, 1950